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NOUVELLE SÉRIE

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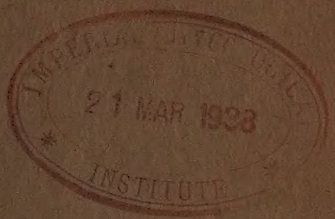

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MATHÉMATIQUES

SUR UNE MÉTHODE DES APPROXIMATIONS SUCCESSIVES

Par N. W. ADAMOFF

(*Présenté par S. N. Bernstein, de l'Académie, le 11. XII. 1937*)

1. Soit

$$y' = f(x, y), \tag{1}$$

une équation différentielle, où $f(x, y)$ est une fonction de x, y, continue dans le domaine $G\,(a \leqslant x \leqslant b,\ c \leqslant y \leqslant d)$ et admettant dans ce domaine une dérivée continue $\frac{\partial f}{\partial y}$. La méthode des approximations successives de Picard permet d'établir l'existence et l'unicité des solutions de l'équation (1). Ces solutions sont des fonctions continues du paramètre y_0, qui présente la valeur de la solution pour $x = a$, et admettent une dérivée continue $\frac{\partial y}{\partial y_0}$ par rapport à la valeur initiale. On se sert ordinairement pour les approximations successives de l'opérateur

$$A[y] = y_0 + \int_a^x f(x, y)\, dx.$$

L'application de cet opérateur exige, en général, une restriction de l'intervalle des valeurs de x. On prend

$$a \leqslant x \leqslant a_1 < b.$$

Nous nous posons ici le problème de trouver un opérateur $A[y]$ tel que 1) la recherche de la fonction $y(x)$ vérifiant l'égalité

$$y = A[y],$$

est équivalente à la résolution de l'équation (1); 2) l'operateur $A[y]$ a un sens dans tout l'intervalle $a \leqslant x \leqslant b$, 3) si la famille de courbes

$$y = \varphi(x, y_0)$$

forme un champ dans le domaine G, c'est-à-dire, si deux courbes différentes de la famille n'ont pas de point commun, les courbes

$$y = A[\varphi(x, y_0)]$$

forment aussi un champ dans le domaine G; 4) l'opérateur $A[y]$ est continu dans le domaine G, c'est-à-dire, étant donnée une fonction $y(x)$ telle que les courbes

$$y = y(x); \quad y = A[y(x)]$$

sont situées dans le domaine G pour $a \leqslant x \leqslant b$, et un nombre positif arbitraire ε, on peut trouver un nombre positif δ tel que l'inégalité

$$|y(x) - \bar{y}(x)| < \delta$$

a pour conséquence l'inégalité

$$|A[y(x)] - A[\bar{y}(x)]| < \varepsilon.$$

2. Nous allons démontrer que l'on peut trouver l'opérateur $A[y(x)]$ correspondant à l'équation (1), satisfaisant aux conditions 1), 2), 3), 4) du paragraphe précédent. Cherchons cet opérateur dans la forme

$$A[y(x)] = y(x) + \int_a^x (Y\,dx - X\,dy),$$

où Y et X sont deux fonctions de x et y, telles que

$$Y = X f(xy).$$

On voit que la condition 1) est satisfaite, si l'on a $X > 0$. En effet, l'égalité

$$y(x) = A[y(x)]$$

n'est vérifiée que sous la condition:

$$Y\,dx - X\,dy = 0,$$

c'est-à-dire:

$$y' = f(x, y).$$

Si la fonction $y(x)$ vérifie l'inégalité

$$c < y(x) < d,$$

on peut multiplier X et Y par un facteur constant positif que l'on peut prendre aussi petit que l'on veut. En opérant de cette manière on peut prendre la valeur absolue de l'intégrale:

$$\int_a^x Y\,dx - X\,dy$$

aussi petite que l'on veut, et la courbe

$$y = A[y(x)]$$

sera située dans le domaine G.

Nous dirons que l'opérateur A a un sens pour une fonction $y(x)$, si l'intégrale

$$\int_a^x Y\,dx - X$$

existe quel que soit x dans l'intervalle $a \leqslant x \leqslant b$, et si la courbe $y = A[y(x)]$ est située dans le domaine G.

Il faut maintenant vérifier la condition 3). Nous désignerons par C une famille de fonctions $y(x, y_0)$ admettant dans le domaine G des dérivées premières par rapport à x et y_0 et telles que la dérivée $\frac{\partial^2 y}{\partial x \partial y_0}$ existe dans le domaine G.

La condition 3) sera satisfaite si l'inégalité

$$\frac{\partial y}{\partial y_0} > 0$$

a pour conséquence l'inégalité

$$\frac{\partial}{\partial y_0}\Big(A\,[y(x, y_0)]\Big) > 0$$

Or

$$\frac{\partial}{\partial y_0} A\,[y(x, y_0)] = \frac{\partial y}{\partial y_0} + \int_a^x \Big(\frac{\partial Y}{\partial y}\frac{\partial y}{\partial y_0} - \frac{\partial X}{\partial y}\frac{\partial y}{\partial y_0} - X\frac{\partial^2 y}{\partial x \partial y_0}\Big)\,dx,$$

en intégrant par parties le dernier terme entre les parenthèses on a:

$$\frac{\partial}{\partial y_0} A\,[y(x, y_0)] = \frac{\partial y}{\partial y_0}(1 - X) + \Big(X\frac{\partial y}{\partial y_0}\Big)_{x=a} + \int_a^x \frac{\partial y}{\partial y_0}\Big(\frac{\partial Y}{\partial y} + \frac{\partial X}{\partial x}\Big)\,dx.$$

On voit que si la fonction X vérifie les inégalités:

$$X > 0 \ (2); \quad X < 1 \ (2'); \quad \frac{\partial Y}{\partial y} + \frac{\partial X}{\partial x} > 0 \ (2''),$$

alors l'inégalité

$$\frac{\partial y}{\partial y_0} > 0$$

a pour conséquence

$$\frac{\partial}{\partial y_0} A\,[y(x, y_0)] > 0,$$

et la condition 3) est satisfaite pour toutes les familles de fonctions analogues à la famille C. On peut prendre $X = \varphi(x)$, et l'inégalite (2″) se réduit à

$$\varphi'(x) + \varphi(x)\frac{\partial f}{\partial y} > 0,$$

ou, en supposant $\varphi(x)$ positive,

$$\frac{\varphi'(x)}{\varphi(x)} > \frac{\partial f}{\partial y};$$

la dérivée $\frac{\partial f}{\partial y}$, étant bornée, $\left|\frac{\partial f}{\partial y}\right| < M$, on peut prendre

$$\varphi(x) = he^{M(x-a)},$$

et l'inégalité (2″) est satisfaite. On satisfait aux conditions (2) et (2′) par un choix convenable de la constante $h > 0$.

Maintenant il est facile de démontrer que l'opérateur A satisfait à la condition 3) par rapport à une famille arbitaire de fonctions $y(x, y_0)$ telles que les courbes $y = y(x, y_0)$ forment un champ D dans le domaine G, et l'opérateur A a un sens pour les fonctions de la famille.

Prenons deux courbes du champ D:

$$y = y(x, y_0') \quad \text{et} \quad y = y(x, y_0'') \quad (y_0' > y_0'').$$

Les courbes $y = A\,[y(x, y_0')]$ et $y = A\,[y(x, y_0'')]$ n'ont pas de point commun. Formons un champ auxiliaire D':

$$y = y(x, y_0'') + \frac{y_0 - y_0''}{y_0' - y_0''}\Big(y(x, y_0') - y(x, y_0'')\Big).$$

Ce champ est analogue au champ C dont nous venons de parler. L'opérateur A a un sens pour les fonctions de la famille D' et il transforme le champ D' en un champ nouveau D''. Les deux courbes de D'', correspondant aux valeurs y_0' et y_0'' du paramètre, c'est-à-dire, les courbes: $y = A[y(x, y_0')]$ et $y = A[y(x, y_0'')]$, n'ont pas de point commun. Ces deux courbes sont des courbes arbitraires du champ D, donc l'opérateur A transforme le champ D en un champ nouveau.

Pour démontrer la continuité de l'opérateur A il suffit de présenter la différence $A[y(x)] - A[\bar{y}(x)]$ sous la forme:

$$y(x) - \bar{y}(x) + \int_a^x \Big(Y(x, y) - Y(x, \bar{y})\Big) dx -$$

$$- \Big(X(y - \bar{y})\Big)_a^x + \int_a^x \frac{dX}{dx}(y - \bar{y})\, dx,$$

la continuité de l'opérateur est une conséquence de la continuité des fonctions $Y(x, y), X(x)$ $\frac{dX}{dx}$.

S'il existe une solution $\bar{y}(x)$ de l'équation (1), égale à y_0 pour $x = a$ et telle que la courbe $y = \bar{y}(x)$, est située dans le domaine G, et si l'opérateur A a un sens pour les fonctions

$$y_0(x), \quad y_1(x) = A[y_0(x)], \quad y_2(x) = A[y(x)], \ldots,$$

où $y_0(x)$ est une fonction égale à y_0 pour $x = a$, la suite: $y_0(x), y_1(x), \ldots$ converge uniformément vers $\bar{y}(x)$.

En effet, on a pour n positif entier:

$$|y_n(x) - \bar{y}(x)| < M\Big[(1-k)^n + n(1-k)^{n-1} Bl(x-a) +$$

$$+ C_n^2 (1-k)^{n-2} \frac{(Bl(x-a))^2}{2!} + \ldots + \frac{(Bl(x-a))^n}{n!}\Big], \qquad (3)$$

$$k \leqslant X \leqslant l \leqslant 1; \quad |y_0(x) - \bar{y}(x)| < M; \quad \left|2\frac{\partial f}{\partial y}\right| < B$$

(on a vu précédemment que ces nombres existent). Soit h un nombre positif, inférieur à $\frac{k}{2}$. On peut trouver un nombre entier m tel que l'on a

$$\frac{(Bl(x-a))^p}{p!} < h^p$$

pour $p > m$. Le second membre de l'inégalité (3) est la somme de deux termes dont le premier est

$$M\Big[(1-k)^n + n(1-k)^{n-1} Bl(x-a) + \ldots + C_n^m (1-k)^{n-m} \frac{(Bl(x-a))^m}{m!}\Big],$$

le nombre $1-k$ étant inférieur à l'unité, les termes de la somme précédente tendent vers zéro avec $\frac{1}{n}$. Le second terme du second membre de l'inégalité (3)

$$C_n^{m+1}(1-k)^{n-m-1} \frac{(Bl(x-a))^{m+1}}{(m+1)!} + \ldots + \frac{(Bl(x-a))^n}{n!} < \Big(1 - \frac{k}{2}\Big)^n,$$

tend aussi vers zéro avec $\frac{1}{n}$.

3. Les théorèmes suivants donnent quelques propriétés des approximations successives du type considéré.

Théorème I. *Si la fonction initiale* $y_0(x)$ *est supérieure à une intégrale* $\bar{y}(x)$ *de l'équation* (1) *pour toutes les valeurs de* $x\,(a \leqslant x \leqslant b)$, *chacune des fonctions*:

$$y_1(x) = A[y_0(x)], \quad y_2(x) = A[y_1(x)], \ldots,$$

est aussi supérieure à $\bar{y}(x)$ *pour* $a \leqslant x \leqslant b$.

Il suffit évidemment de démontrer notre théorème pour la première approximation. Or, formons le champ des courbes:

$$y = \bar{y}(x) + \frac{y_0 - \beta}{\alpha - \beta}(y_0(x) - \bar{y}(x)),$$

où α et β sont les valeurs des fonctions $y_0(x)$ et $\bar{y}(x)$ pour $x = a$. Ce champ se transforme par l'application de l'opérateur A en un autre champ et la courbe du champ $y = \bar{y}(x)$ reste invariable. Aucune courbe de ce champ nouveau, y compris la courbe $y = y(x) = A[y_0(x)]$, ne possède de point commun avec la courbe $y = \bar{y}(x)$, ce que prouve le théorème.

Note. Le théorème reste valable dans le cas

$$y_0(x) > \bar{y}(x) \text{ pour } a < x \leqslant b; \quad y_0(x) = \bar{y}(a).$$

Nous omettons la démonstration de ce fait, ainsi que la démonstration des deux théorèmes suivants:

Théorème II. *Si l'opérateur* A *a un sens pour les fonctions*

$$y_0(x), \quad y_1(x) = A[y_0(x)], \ldots,$$

et si l'on a

$$y_0(x) > y_1(x), \qquad (a < x \leqslant b),$$

on a aussi

$$y_1(x) > y_2(x) > y_3(x) > \ldots,$$

c'est-à-dire, les approximations successives forment une suite monotone.

Théorème III. *Si* 1) *l'équation* (1) *admet une solution* $\bar{y}(x)$ *telle que la courbe intégrale est située dans le domaine* G *pour* $a \leqslant x \leqslant b$; 2) *l'opérateur* A *a un sens pour la fonction* $y_0(x)$; 3) *les courbes* $y = y_0(x)$ *et* $y = A[y_0(x)]$ *n'ont pas de point commun différent du point initial, la courbe intégrale et la courbe* $y = y_0(x)$ *n'ont pas de point commun différent du point initial.*

La démonstration est toujours la même—celle de formation des champs auxiliaires.

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MATHÉMATIQUES

SUR LA THÉORIE DES SILLAGES

Par M. LAVRENTIEFF

(Présenté par I. M. Vinogradow de l'Académie, le 17. XII. 1937)

Je me propose d'indiquer dans cette note quelques propriétés nouvelles des fonctions univalentes. Ces propriétés permettent de démontrer certains théorèmes d'existence dans la théorie générale des sillages d'un courant potentiel plan.

1. Dans la suite nous désignerons par $y = \varphi(x)\,(y = \varphi_1(x))$ une fonction univoque continue, possédant les dérivées des deux premiers ordres sur $-\infty < x < +\infty$, qui admet des valeurs limites finies quand $x \to \pm\infty$. Nous désignerons par $D(\varphi)\,(D(\varphi_1))$ le domaine du plan (x, y) défini par la condition $y < \varphi(x)\,(y < \varphi_1(x))$, et par $w = f(z, \varphi)$, $f'(\infty, \varphi) = 1$, la fonction qui réalise la représentation conforme du domaine $D(\varphi)$ sur le demi-plan inférieur $v < 0$, $w = u + iv$. Dans cette notation nous avons les propositions suivantes.

Lemme. Si

$$\varphi_1(x) < \varphi(x) \quad \text{pour} \quad |x - x_0| < h,$$
$$\varphi_1(x) \equiv \varphi(x) \quad \text{pour} \quad |x - x_0| \geqslant h,$$

on a le long de la ligne $y = \varphi(x)$, $|x - x_0| > h$, la relation

$$|f'(z, \varphi_1)| < |f'(z, \varphi)|.$$

D'ailleurs, en posant pour $y = \varphi(x)$

$$\left|\frac{f'(z, \varphi_1)}{f'(z, \varphi)}\right| = \alpha(x),$$

nous aurons (pour $\Delta x > 0$):

$$\alpha(x + \Delta x) > \alpha(x), \text{ si } x \geqslant x_0 + h,$$
$$\alpha(x - \Delta x) > \alpha(x), \text{ si } x \leqslant x_0 - h.$$

Théorème 1. Supposons qu'un cercle K du rayon r est situé à l'extérieur (à l'intérieur) du domaine $D(\varphi)$ et que sa circonférence a un arc φ commun avec la courbe γ, on a alors le long de γ:

$$\frac{d^2}{ds^2}|f'(z, \varphi)| < 0 \quad \left(\frac{d^2}{ds^2}|f'(z, \varphi)| > 0\right) \tag{1}$$

où ds est l'élément de l'arc de la courbe φ. Si, d'ailleurs, la courbure de la courbe φ ne surpasse pas $\frac{1}{r}$ au voisinage des extrémités de γ, la

fonction $|f'(x+i\varphi(x),\ \varphi)|$ ne peut atteindre son minimum (maximum) aux extrémités de γ.

Théorème 1'. Si la courbe φ est analytique et si en un point A sa courbure atteint son maximum, on a en ce point l'une des inégalités (1), suivant que la courbe φ tourne (au point A) sa convexité ou sa concavité vers le domaine D.

Théorème 2. Supposons qu'une droite $y = kx + b$, $k > 0$ a un segment δ commun avec la courbe, φ et qu' à droite de δ la courbe φ est située au dessous de la droite tandis qu'à gauche elle est située au dessus de cette droite. Dans ces conditions on a le long de δ:

$$\frac{d}{dx}|f'(x+i\varphi(x),\ \varphi)| < 0, \qquad (2)$$

et d'ailleurs la fonction $|f'(x+i\varphi(x),\ \varphi)|$ ne peut atteindre ni son maximum ni son minimum aux extrémités du segment δ.

Théorème 2'. Si la courbe φ est analytique et si au point B la dérivée $\varphi'(x)$ atteint son maximum, on a au point B l'inégalité (2).

2. En s'appuyant sur les propositions qui viennent d'être énoncées et en applicant les méthodes directes du calcul des variations, on peut démontrer les théorèmes suivants de la théorie des sillages.

Théorème 3. Quelle que soit la fonction univoque $y = \psi_1(x)$ qui pour $x \leqslant 0$ possède une seconde dérivée bornée et telle que

$$\lim_{x \to -\infty} \psi_1(x) = 0\ *,$$

il existe toujours une et une seule fonction $y = \psi_2(x)$, $x \geqslant 0$, $\psi_2(0) = \psi_1(0)$, telle que le long de la courbe ψ_2 on a:

$$|f'(z, \varphi)| = 1,$$

où $\varphi(x) \equiv \psi_1(x)$ pour $x \leqslant 0$ et $\varphi \equiv \psi_2$ pour $x > 0$. Si, d'ailleurs, ψ_1 est non positive et admet sa plus petite valeur pour $x = 0$, la fonction cherchée ψ_2 est toujours négative **.

Théorème 4. Dans les conditions du théorème 3, si ψ_1 est non positive, $\psi_1(0) < 0$, et ψ_2 admet des valeurs positives, il existe une et une seule fonction $y = \psi_3(x)$, $\psi_3(0) = \psi_1(0)$ négative dans un certain intervalle $(0, h)$, $h > 0$, nulle pour $x \geqslant h$ et telle que le long de ψ_3, $0 \leqslant x \leqslant h$ on a:

$$|f'(z, \varphi)| = c = \text{const},$$

ou $\varphi \equiv \psi$ pour $x \leqslant 0$ et $\varphi \equiv \psi_3$ pour $x > 0$.

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* Sans restreindre la généralité on peut supposer que la courbe ψ_1 est formée d'un nombre fini de morceaux de lignes

$$y = \chi(x),\ |\chi''(x)| \leqslant M = \text{const}.$$

** La résolution la plus générale du problème sur l'existence des sillages a été donnée par Leray au moyen de la méthode des équations intégrales. Dans la théorie de Leray on suppose que chaque parallèle à l'axe de x ne coupe qu'en un seul point l'arc qui se trouve dans le courant.

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MATHÉMATIQUES

SUR LES CONDITIONS POUR QUE LE PROBLÈME DE CAUCHY POUR UN SYSTÈME D'ÉQUATIONS AUX DÉRIVÉES PARTIELLES DU PREMIER ORDRE SOIT CORRECTEMENT POSÉ

Par S. HALPERN

(Présenté par S. N. Bernstein, de l'Académie, le 20. XII. 1937)

I. Petrowsky a posé le problème de trouver les conditions pour que pour un système d'équations aux dérivées partielles dont le nombre d'équations est supérieur au nombre de fonctions inconnues, le problème de Cauchy soit correctement posé. Nous envisageons ce problème dans le cas d'un système d'équations aux dérivées partielles du premier ordre, tel que 1) le nombre q d'équations est un multiple du nombre p de fonctions inconnues: $q = mp$; 2) les équations sont résolues par rapport aux dérivées de toutes les fonctions inconnues, prises par rapport à m variables indépendantes, les mêmes pour chaque fonction (voir le système (I)); 3) toutes les variables, les fonctions et les seconds membres des équations sont réels.

1. En désignant par $t_1, t_2, \ldots, t_m, x_1, x_2, \ldots, x_n$ les $n+m$ variables indépendantes, par $z_1, z_2, \ldots, z_p$ les p fonctions inconnues et en posant $p_j^k = \frac{\partial z_j}{\partial x_k}$; $j = 1, 2, \ldots, p$; $k = 1, 2, \ldots, n$, on peut écrire le système considéré sous la forme:

$$\left.\begin{aligned}
&\frac{\partial z_i}{\partial t_1} = F_i^{(1)}(t_j, x_k, z_l, p_s^r), && (\mathrm{I}, 1),\\
&\frac{\partial z_i}{\partial t_2} = F_i^{(2)}(t_j, x_k, z_l, p_s^r), && (\mathrm{I}, 2),\\
&\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\\
&\frac{\partial z_i}{\partial t_m} = F_i^{(m)}(t_j, x_k, z_l, p_s^r) && (\mathrm{I}, m),\\
&\qquad\qquad i = 1, 2, \ldots, p.
\end{aligned}\right\} \qquad (\mathrm{I})$$

le nombre d'équations dans ce système est égal à mp; les $F_i^{(k)}$, $i = 1, 2, \ldots, p$; $k = 1, 2, \ldots, m$, sont des fonctions de tous les t_j, x_k, z_l, et p_s^r; $j = 1, 2, \ldots, m$; $k = 1, 2, \ldots, n$; $l = 1, 2, \ldots, p$; $r = 1, 2, \ldots, n$; $s = 1, 2, \ldots, p$; on suppose parfois qu'elles admettent des dérivées par-

* Cet ordre sera précisé dans la suite.

tielles jusqu'à un ordre suffisamment élevé * et parfois qu'elles sont analytiques.

Soit H le parallélépipède

$$0 \leqslant t_i \leqslant T_i; \qquad |x_j| \leqslant a; \qquad (2)$$
$$i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n;$$

et D le domaine

$$0 \leqslant t_i \leqslant T_i; \qquad |x_j| \leqslant a; \qquad |z_l - \overline{\varphi_l(x_k)}| < M;$$

$$\left| p_s^r - \overline{\frac{\partial \varphi_s}{\partial x_r}} \right| < M_1, \qquad (3)$$

$$i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n; \quad l = 1, 2, \ldots, p;$$
$$s = 1, 2, \ldots, p; \quad r = 1, 2, \ldots, n.$$

Nous dirons que les équations du système (I) *sont compatibles dans le domaine D, lorsque pour tout système de fonctions* $\varphi_i(x_k)$; $i = 1, 2, \ldots, p$, *appartenant au domaine D. et pour tout système de nombres* $(t_j)_0$ *tels que* $0 \leqslant (t_j)_0 \leqslant T_j$, $j = 1, 2, \ldots, m$, *il existe des fonctions* $z_i = f_i(t_j, x_k)$, $i = 1, 2, \ldots, p$, *qui satisfont aux équations du système* (I) *et qui se réduisent pour* $(t_j) = (t_j)_0$; $j = 1, 2, \ldots, m$ *aux fonctions* $\varphi_i(x_k)$.

Nous supposons d'ailleurs

a) que les solutions $f_i(t_j, x_k)$ existent dans une bande du parallélépipède H: $|t_j - (t_j)_0 < \delta$; $j = 1, 2, \ldots, m$ qui peut être, d'ailleurs, aussi petite que l'on veut;

b) que les fonctions initiales $\varphi_i(x_k)$ et $\overline{\varphi}_i(x_k)$ et les seconds membres $F_i^{(j)}$ des équations du système (I) admettent des dérivées jusqu'à un ordre suffisamment élevé ou bien qu'ils sont analytiques *.

Considérons les matrices

$$A_l^k = \left\| \begin{matrix} \frac{\partial F_1^{(l)}}{\partial p_1^k} & \frac{\partial F_1^{(l)}}{\partial p_2^k} & \cdots & \frac{\partial F_1^{(l)}}{\partial p_p^k} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial F_p^{(l)}}{\partial p_1^k} & \frac{\partial F_p^{(l)}}{\partial p_2^k} & \cdots & \frac{\partial F_p^{(l)}}{\partial p_p^k} \end{matrix} \right\|, \qquad \begin{matrix} l = 1, 2, \ldots, m, \\ k = 1, 2, \ldots, n. \end{matrix}$$

Les conditions de compatibilité du système (I) dans le domaine D nous donnent les relations:

$$A_l^k A_r^j + A_l^j A_r^k = A_r^k A_l^j + A_r^j A_l^k, \qquad (4)$$
$$k, j = 1, 2, \ldots, n;$$
$$r, l = 1, 2, \ldots, m,$$

qui sont valables pour tout système de valeurs des variables appartenant à D. Posons $A_l = \sum_{k=1}^{n} A_l^k \alpha_k$, les α_k étant des paramètres réels arbitraires,

---

* Les seconds membres et les fonctions initiales sont supposés analytiques dans le cas où pour l'un des systèmes (I, i) le théorème d'existence est établi seulement sous l'hypothèse que les $F_i^{(j)}$ et les φ_i sont analytiques.

satisfaisant à la relation $\sum_{k=1}^{n} \alpha_k^2 = 1$; on peut alors remplacer les relations (4) par des relations équivalentes:

$$A_l A_r = A_r A_l, \qquad (5)$$
$$l, r = 1, 2, \ldots, m.$$

Donc, si le système (I) *est compatible dans le domaine* D, *les matrices* A_l *et* A_r; $l, r = 1, 2, \ldots, m$ *sont commutatives pour toutes les valeurs des* t_j, x_k, z_l, p_s^r, *et pour toutes les valeurs réelles des* α_k *telles que* $\sum_{k=1}^{n} \alpha_k^2 = 1$.

2. I. Petrowsky en prenant pour point de départ les idées de J. Hadamard a défini [1] ce qu'il faut entendre par un problème de Cauchy correctement posé. En appliquant cette définition au système (I), nous dirons que le problème de Cauchy pour ce système est correctement posé pour les fonctions initiales $\overline{\varphi_i}(x_k)$; $i = 1, 2, \ldots, p$ et les valeurs initiales $t_j = (t_j)_0$; $j = 1, 2, \ldots, m$, lorsque les conditions suivantes se trouvent vérifiées:

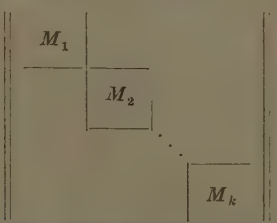
1) Pour tout système de fonctions $\varphi_i(x_k)$ admettant des dérivées jusqu'à un certain ordre L et tel que les fonctions $\varphi_i(x_k)$ ainsi que leurs dérivées jusqu'à l'ordre L diffèrent suffisamment peu des fonctions $\overline{\varphi_i}(x_k)$ et de leurs dérivées, il existe un système unique de fonctions $z_i = f_i(t_j, x_k)$ qui satisfont au système (I) dans un domaine D_1 contenu dans D et adjacent à l'hyperplan $t_j = (t_j)_0$, et qui se réduisent pour $t_j = (t_j)_0$ aux fonctions données $\varphi_i(x_k)$.

2) A tout nombre $\varepsilon > 0$ on peut assigner un nombre $\eta > 0$ de manière que lorsque les valeurs absolues des différences entre les fonctions $\varphi_i(x_k)$ et $\overline{\varphi_i}(x_k)$ ainsi que de celles entre leurs dérivées jusqu'à l'ordre L sont inférieures à η. alors les solutions correspondant aux fonctions initiales $\varphi_i(x_k)$ diffèrent des solutions correspondant aux fonctions initiales $\overline{\varphi_i}(x_k)$ moins de ε dans le domaine D_1.

Le nombre L et le domaine D_1 sont supposés indépendants du choix de ε.

I. Petrowsky [1] a démontré que pour un système dans lequel le nombre de fonctions inconnues et égal au nombre d'équations [donc pour un système (I, 1) considéré séparement] le problème de Cauchy est correctement posé, si c'est un système hyperbolique ou un système hyperbolique généralisé *.

* Un système (I. 1) est hyperbolique lorsque 1) la matrice $A_1 - \lambda E$ est de la forme



tous les éléments situés hors des carrés M_i étant nuls pour toutes les valeurs admissibles des variables; 2) les racines de chacune des équations $|M_i| = 0$ sont réelles et différentes (leur différence étant $\geqslant \delta > 0$).

Un système est un système hyperbolique généralisé lorsqu'il existe une transformation T réelle et non singulière (le déterminant de T étant $\geqslant \delta > 0$), qui réduit la matrice $A_1 - \lambda E$ à la forme diagonale.

En s'appuyant sur le fait que les matrices A_l et A_r d'un système compatible sont commutatives et en appliquant le théorème déjà cité de I. Petrowsky, on obtient le théorème suivant:

Si le système (1) *est compatible dans le domaine D et si les racines de l'équation caractéristique de l'une des matrices* A_i (*par exemple* A_1) *sont réelles et différentes* (*leur différence étant* $\geqslant \delta > 0$) *partout dans le domaine D et pour toutes les valeurs réelles des* α_k *telles que* $\sum_{k=1}^{n} \alpha_k^2 = 1$, *alors le problème de Cauchy est correctement posé dans le voisinage des fonctions* $\varphi_i(x_k)$.

On suppose que les fonctions initiales et les seconds membres des équations admettent des dérivées jusqu'à l'ordre

$$(n+3)(m-1)+4n+8.$$

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MATHEMATIK

ÜBER DIE KONVERGENZ DER VERTEILUNGSGESETZE VON SUMMEN VONEINANDER UNABHÄNGIGER SUMMANDEN

Von B. GNEDENKO

(Vorgelegt von S. N. Bernstein, Mitglied der Akademie, am 17. XII. 1937)

Es sei eine Folge von Serien zufälliger Grössen:

$$x_{n,1},\ x_{n,2},\ \dots,\ x_{n,k_n}\ (n = 1, 2, \dots), \tag{1}$$

gegeben, welche in jeder Serie voneinander unabhängig und «im Limes» vernachlässigbar» sind, d. h. derart, dass in bezug auf k $(1 \leqslant k \leqslant k_n)$ gleichmässig

$$\lim_{n\to\infty} P\{|x_{n,k}| > \varepsilon\} \to 0$$

für beliebiges $\varepsilon > 0$.

Die Verteilungsgesetze der zufälligen Grössen (1) bezeichnen wir entsprechend mit

$$F_{n,1}(x),\ F_{n,2}(x), \dots,\ F_{n,k_n}(x).$$

Für die charakteristische Funktion des Gesetzes $F_{n,k}(x)$ führen wir die Bezeichnung:

$$f_{n,k}(t) = \rho_{n,k}(t)\, e^{i\omega_{n,k}(t)},$$

ein, worin $\rho_{n,k}(t) \geqslant 0$ und $\omega_{n,k}(t)$ reell sind, $-\pi \leqslant \omega_{n,k}(t) < \pi$.

Da die $x_{n,k}$ im Limes vernachlässigbar sind, so gilt

$$\rho_{n,k}(t) \to 1,\ \omega_{n,k}(t) \to 0 \text{ für } n \to \infty,$$

gleichmässig in bezug auf $k(1 \leqslant k \leqslant k_n)$ und t in jedem endlichen Intervall $(|t| \leqslant T)$.

Es lässt sich folgender Satz beweisen.

Satz. I. *Damit die Folge von Verteilungsgesetzen $F_n(x)$ der Summen:*

$$s_n = x_{n,1} + x_{n,2} + \dots + x_{n,k_n}, \tag{2}$$

bei $n \to \infty$ gegen ein Limesgesetz $\Phi(x)$ konvergiert, ist notwendig und hinreichend, dass dasselbe für die Folge der unbeschränkt teilbaren Gesetze $\Phi_n(x)$ zutrifft, deren charakteristische Funktionen $\varphi_n(t)$ durch

$$\lg \varphi_n(t) = \sum_{k=1}^{k_n} \left\{ i\omega_{n,k}(1)\, t + \int_{-\infty}^{+\infty} (e^{itx} - 1)\, d F_{n,k}[x + \omega_{n,k}(1)] \right.$$

definiert sind.

Die Grenzgesetze der beiden Folgen fallen dabei zusammen.

Unter der Voraussetzung, dass die quadratischen Abweichungen der Summen beschränkt sind, wurde ein analoger Satz von G. M. Bawly (1) bewiesen. Nach Satz I ist es möglich, in der Theorie der Limesgesetze von Summen unabhängiger zufälliger Grössen sich auf die Betrachtung der unbeschränkt teilbaren Gesetze zu beschränken. Die Vermutung, dass die Summen von im Limes vernachlässigbaren Summanden durch unbeschränkt teilbare Grössen approximiert werden können, wurde mehrmals von A. Kolmogoroff ausgesprochen.

Aus Satz I lässt sich unmittelbar das folgende, von A. Khintchine (2) bewiesene Resultat ableiten:

Die Klasse der Limesgesetze für Summen (2) *voneinander unabhängiger, im Limes vernachlässigbarer zufälliger Grössen fällt mit der Klasse der unbeschränkt teilbaren Gesetze zusammen.*

Damit das Gesetz $\Phi(x)$ unbeschränkt teilbar sei, ist bekanntlich notwendig und hinreichend, dass der Logarithmus seiner charakteristischen Funktion sich durch die Formel von P. Lévy:

$$\lg \varphi(t) = i\gamma t + \int_{-\infty}^{+\infty} \left\{ e^{iut} - 1 - \frac{iut}{1+u^2} \right\} \frac{1+u^2}{u^2}\, dG(u),$$

darstellen lässt, worin γ eine Konstante und $G(u)$ eine nicht abnehmende beschränkte Funktion bedeutet.

Bezüglich der Konvergenz einer Folge von unbeschränkt teilbaren Gesetzen gilt der folgende

Satz II. *Damit eine Folge* $\{\Phi_n(x)\}$ *unbeschränkt teilbarer Gesetze gegen ein Limesgesetz* $\Phi(x)$ *konvergiert, ist notwendig und hinreichend, dass für* $n \to \infty$:

1) $G_n(u) \to G(u)$, in jedem Stetigkeitspunkte von $G(u)$,
2) $\underset{-\infty<u<\infty}{\text{var}}\, G_n(u) \to \underset{-\infty<u<\infty}{\text{var}}\, G(u)$,
3) $\gamma_n \to \gamma$,

wobei γ, $G(u)$, γ_n, $G_n(u)$ nach der Formel von P. Lévy für die Gesetze $\Phi(x)$ und $\Phi_n(x)$ definiert werden.

Als Anwendung der Sätze I und II wollen wir die notwendigen und hinreichenden Bedingungen aufstellen, unter welchen die Verteilungsgesetze der Summen (2) gegen das Gauss'sche Gesetz konvergieren.

Auf Grund eines Satzes von H. Cramer (3) können wir uns dabei auf die Betrachtung symmetrisch verteilter Summanden $x_{n,k}$ beschränken. Wegen Satz I gilt

$$F_n(x) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\, dt \qquad (n \to \infty)$$

dann und nur dann, wenn

$$\sum_{k=1}^{k_n} \int_{-\infty}^{+\infty} (\cos xt - 1)\, dF_{n,k}(x) \to -\frac{t^2}{2} \qquad (n \to \infty)$$

ist. Nach Satz II ist dazu notwendig und hinreichend, dass gleichzeitig

$$G_n(x) = \sum_{k=1}^{k_n} \int_{-\infty}^{x} \frac{u^2}{1+u^2}\, dF_{n,k}(x) \underset{(n\to\infty)}{\to} G(x) = \begin{cases} 0 & (x<0), \\ 1 & (x>0), \end{cases}$$

und

$$G_n(+\infty) \to 1 \qquad (n \to \infty)$$

gilt. Diese Bedingungen sind den folgenden äquivalent:

$$\sum_{k=1}^{k_n} \int_{|x|>\varepsilon} \frac{x^2}{1+x^2}\, dF_{n,k}(x) \underset{n\to\infty}{\to} 0, \tag{3}$$

$$\sum_{k=1}^{k_n} \int_{|x|<\varepsilon} \frac{x^2}{1+x^2}\, dF_{n,k}(x) \underset{n\to\infty}{\to} 1, \tag{4}$$

für beliebiges $\varepsilon > 0$.

Infolge der für $u > \varepsilon$ gültigen Ungleichung $\frac{u^2}{1+u^2} > \frac{\varepsilon^2}{1+\varepsilon^2}$ überzeugt man sich, dass die Relation (3) mit

$$\sum_{k=1}^{k_n} \int_{|x|>\varepsilon} dF_{n,k}(x) \underset{n\to\infty}{\to} 0 \tag{5}$$

äquivalent ist. Aus der evidenten Gleichung

$$\sum_{k=1}^{k_n} \int_{|x|<\varepsilon} \frac{x^2}{1+x^2}\, dF_{n,k}(x) = \sum_{k=1}^{k_n} \int_{|x|<\varepsilon} x^2 dF_{n,k}(x) - \sum_{k=1}^{n} \int_{|x|<\varepsilon} \frac{x^4}{1+x^2}\, dF_{n,k}(x)$$

und der Formel (5) ziehen wir den Schluss, dass (4) mit der Relation

$$\sum_{k=1}^{k_n} \int_{|x|<\varepsilon} x^2 dF_{n,k}(x) \underset{n\to\infty}{\to} 1 \tag{6}$$

äquivalent ist.

Die Relationen (5) und (6) liefern eine Verallgemeinerung der von W. Feller (4) aufgestellten Bedingungen, unter welchen die Verteilungsgesetze von Summen

$$s_n = \frac{x_1 + x_2 + \ldots + x_n}{B_n} - A_n$$

der n ersten Glieder einer Folge $x_1, x_2, \ldots, x_n, \ldots$ unabhängiger zufälliger Grössen, bei passender Wahl der Konstanten A_n und $B_n > 0$, gegen das Gauss'sche Gesetz konvergieren.

Mit Hilfe analoger Überlegungen lässt sich beweisen, dass für die Konvergenz der Verteilungsgesetze von Summen (2) gegen das Poissonsche Gesetz $P(x) = \sum_{0 \leqslant k \leqslant x} \frac{e^{-1}}{k!}$ die Erfüllung der folgenden Bedingungen notwendig und hinreichend ist:

1) $\sum_{k=1}^{k_n} \int_{x>1+\varepsilon} dF_{n,k}(x) \to 0 \qquad (n \to \infty)$

2) $\sum_{k=1}^{k_n} \int_{|x-1|<\varepsilon} dF_{n,k}(x) \to 2 \qquad (n \to \infty)$

3) $\sum_{k=1}^{k_n} \int_{x<-1+\varepsilon} dF_{n,k}(x) \to 0 \qquad (n \to \infty)$

4) $\sum_{k=1}^{k_n} \int_{|x|<1-\varepsilon} x\, dF_{n,k}(x) \to 0 \qquad (n \to \infty)$

5) $\sum_{k=1}^{k_n} \int_{|x|<1-\varepsilon} x^2 dF_{n,k}(x) \to 0 \qquad (n \to \infty)$

für beliebiges $\varepsilon > 0$.

Aus den Sätzen I und II folgern wir den

Satz III. *Notwendig und hinreichend für die Konvergenz der Verteilungsgesetze der Summen* (2) *gegen ein Limesgesetz ist die Existenz einer Funktion* $G(u)$ *und einer Konstante* γ *derart, dass*

$$1)\quad \sum_{k=1}^{k_n} \int_{-\infty}^{u} \frac{x^2}{1+x^2}\, dF_{n,k}\{x+\omega_{n,k}(1)\} \underset{n\to\infty}{\to} G(u)$$

in jedem Stetigkeitspunkte von $G(u)$,

$$2)\quad G(-\infty)=0,\ G(+\infty)=\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{-\infty}^{+\infty} \frac{x^2}{1+x^2}\, dF_{n,k}\{x+\omega_{n,k}(1)\} < +\infty,$$

$$3)\quad \sum_{k=1}^{k_n} \{\omega_{n,k}(1) + \int_{-\infty}^{+\infty} \frac{x^2}{1+x^2}\, dF_{n,k}\{x+\omega_{n,k}(1)\} \underset{n\to\infty}{\to} \gamma.$$

Der Logarithmus der charakteristischen Funktion des Limesgesetzes wird dabei durch die Formel:

$$\lg \varphi(t) = i\gamma t + \int_{-\infty}^{+\infty} \left\{ e^{iut} - 1 - \frac{iut}{1+u^2} \right\} \frac{1+u^2}{u^2}\, dG(u),$$

gegeben, worin die Funktion $F(u)$ *und die Konstante* γ *mittels der Bedingungen 1—3 dieses Theorems definiert werden.*

Die Sätze I — III können auch zur Aufstellung der Bedingungen für die Anwendbarkeit des Gesetzes der grossen Zahlen dienen.

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MATHÉMATIQUES

SUR CERTAINES ÉVALUATIONS POUR LES FAMILLES DE FONCTIONS VÉRIFIANT QUELQUES INÉGALITÉS INTÉGRALES

Par V. KONDRACHOV

(Présenté par I. M. Vinogradow, de l'Académie, le 17. X. 1937.)

Considérons dans un domaine D d'un espace à n dimensions la famille de fonctions continues ainsi que leurs dérivées d'ordre k par rapport aux arguments $x_1, x_2, \dots x_3$, et vérifiant la condition

$$D(\varphi)' = \int_D^n \cdots \int \sum_{l=0}^{k} \sum_{\Sigma\alpha=l} \left| \frac{\partial^l \varphi}{\partial x_1^{\alpha_1} \dots \partial x_s^{\alpha_s}} \right|^p dx_1\, dx_2 \dots dx_n \leqslant A, \qquad (0,1)$$

où φ est une fonction quelconque de la famille, A — une constante.

Considérons le plan $x_i = 0$ $(i = 1, 2 \dots s)$ dont l'intersection avec le domaine D précédemment défini est une multiplicité F intérieure à D. On définit avec F une multiplicité voisine:

$$x_i = \chi_i(x_{s+1} \dots x_n); \quad |\chi_i| \leqslant \delta, \quad (i = 1, 2 \dots s). \qquad (0,2)$$

En posant

$$I_{n-s}(\varphi) = \frac{1}{|F|} \int_F^{n-s} \cdots \int |\varphi(0, 0 \dots x_{s+1} \dots x_n)|^p\, dx_{s+1} \dots dx_n,$$

$$\Delta(\varphi) = \frac{1}{|F|} \int_F^{n-s} \cdots \int |\varphi(\chi_1\ \chi_2 \dots \chi_s,\ x_{s+1} \dots x_n) -$$

$$- \varphi(0, 0 \dots x_{s+1} \dots x_n)|^p dx_{s+1} \dots dx_n,$$

on peut faire l'hypothèse suivante: si $k > \frac{s}{p}$ et $p > 1$:

$$I_{n-s}(\varphi) \leqslant A \cdot B_1, \qquad (1)$$

$\Delta(\varphi)$ tend vers zéro avec δ.

Plus précisément: si $k = \frac{s+m}{p}$, $m > 0$, on a les trois cas suivants:

I. $m < p$ $\quad \Delta(\varphi) \leqslant A \cdot B_2\, \delta^m$,
II. $m = p$ $\quad \Delta(\varphi) \leqslant A \cdot B_3\, \delta^p |\lg \delta|^{p-1}$, $\quad B_i = \text{const}$
III. $m > p$ $\quad \Delta(\varphi) \leqslant A \cdot B_4\, \delta^p$,

Les constantes B_i sont introduites pour pouvoir appliquer le théorème aux multiplicités qui deviennent planes quand on fait une transformation convenable qui est continue ainsi qu'un nombre suffisant de ses dérivées.

On pourra obtenir quelques résultats supplémentaires liés avec l'énoncé du théorème; nous les indiquerons comme remarques après avoir démontré que les deux propositions précédemment formulées sont vraies.

Le théorème en question est une généralisation d'un théorème de S. L. Soboleff [1]

S. L. Soboleff a étudié le cas où $p = 2$; dans ce cas, si s est impair, on obtient le cas I, et, si s est pair, le cas II.

La méthode que nous suivons est aussi une généralisation d'une méthode que S. L. Soboleff a employé dans son article précédemment indiqué[1].

I. Démonstration de l'existence des bornes.

Après avoir introduit dans l'espace $x_1\, x_2 \dots x_n$ des coordonnées cylindriques d'ordre s ainsi qu'il a été fait par S. L. Soboleff, démontrons l'identité:

$$\varphi(0, 0 \dots 0,\ x_{s+1} \dots x_n) =$$

$$= C_1 \int_{\substack{\rho_s \leqslant h \\ \theta_{s-1} \leqslant \alpha}}^{S} \dots \int Z(\cos\theta_{s-2} - \cos\alpha)\, \rho_s^{k-s}(h - \rho_s)\, dx_1 \dots dx_s, \qquad (I, 1)$$

$$Z = \left[k(k+1)\,\omega + (-1)^{k-1} \cdot \frac{1}{(k-1)!} \cdot \frac{\partial^k \varphi}{\partial \rho_s^k} \left(k\rho_s^k - h\, \frac{\rho_s^k - h^k}{\rho_s - h} \right) \right],$$

$$\omega = \varphi - \frac{\partial \varphi}{\partial \rho_s} \cdot \frac{\rho_s}{1!} + \frac{\partial^2 \varphi}{\partial \rho_s^2} \cdot \frac{\rho_s^2}{2!} - \dots + (-1)^{k-1} \frac{\partial^{k-1} \varphi}{\partial \rho_s^{k-1}} \cdot \frac{\rho_s^{k-1}}{(k-1)!}.$$

Dans tout ce qui suit nous désignerons par C_i et B_i des constantes qui ne dépendent que de h ou de la forme du domaine. Il suit de la relation (I, 1) que l'on a:

$$|\varphi(0, 0 \dots x_{s+1} \dots x_n)| \leqslant$$

$$\leqslant C_2 \int_{\substack{\rho_s \leqslant h \\ \theta_{s-1} \leqslant \alpha}}^{S} \dots \int \sum_{l=0}^{k} \sum_{\Sigma a = l} \left| \frac{\partial^l \varphi}{\partial x_1^{a_1} \dots \partial x_s^{a_s}} \right| \cdot \rho_s^{k-s}\, dx_1\, dx_2 \dots dx_s. \qquad (I, 2)$$

En appliquant l'inégalité de Hölder et en tenant compte de l'évaluation de l'expression $\left(\sum_{i=1}^{n} x_i \right)^p$; $p \geqslant 1$; $x_i \geqslant 0$, on a:

$$|\varphi(0, 0 \dots x_{s+1} \dots x_n)|^p \leqslant C_3 \int_{\substack{\rho_s \leqslant h \\ \theta_{s-1} \leqslant \alpha}}^{S} \dots \int \sum_{l=0}^{k} \sum_{\Sigma a = l} \left| \frac{\partial^l \varphi}{\partial x_1^{a_1} \dots \partial x_s^{a_s}} \right|^p dx_1 \dots dx_s \times$$

$$\times \left(\int_{\substack{\rho_s \leqslant h \\ \theta_{s-1} \leqslant \alpha}}^{S} \dots \int \rho_s^{qk - qs} \cdot dx_1 \dots dx_s \right)^{\frac{p}{q}}. \qquad (I, 3)$$

Il suit des conditions du théorème que $k > \frac{s}{p}$; donc la seconde intégrale donne une constante qui dépend de h. En étendant l' intégration sur F, on a évidemment, en remplaçant l'inégalité par une autre plus forte,

$$\frac{1}{|F|} \int_{F}^{n-s} \dots \int |\varphi(0, 0 \dots x_{s+1} \dots x_n)|^p\, dx_{s+1} \dots dx_n \leqslant$$

$$\leqslant C \int_{D}^{n} \dots \int \sum_{0}^{k} \sum_{\Sigma a = l} \left| \frac{\partial^l \varphi}{\partial x_1^{a_1} \dots \partial x_s^{a_s}} \right|^p dx_1 \dots dx_n.$$

Remarque 1. Considérons les limites de l'intégrale dans le second membre de l'identité (I, 1). Il suit évidemment de tout ce qui précède la relation (I, 1) que la valeur des fonctions φ au point ayant pour coordonnées $(0, 0 \dots 0, x_{s+1} \dots x_n)$ (point de la multiplicité F) se trouve sur la «pointe» de la figure $\rho_s \leqslant h$ et $\theta_{s-2} \leqslant \alpha$ située dans le plan $x_1 \dots x_s$. L'intégration suivante (sur F) fait la «pointe» glisser par les points de la partie de F qui est intérieure à D. Il en résulte que le théorème s'applique au cas, où F est situé dans D de telle manière qu'il soit possible, en restant toujours à l'intérieur de D, d'atteindre chaque point de F au moyen d'une figure de dimensions finies $\rho_s \leqslant h$ et $\theta_{s-2} \leqslant \alpha$ située dans le plan $x_{s+1} = \text{const} \dots x_n = \text{const}$.

Il est évident qu'en général on ne peut pas prendre pour F une partie frontière contenant des arêtes de rebroussement, exception faite pour le cas où ces arêtes sont dirigées à l'intérieur de D.

Dans la suite nous donnerons le nom de «secteur atteignant S» à la figure de dimensions finies

$$\rho_s \leqslant h, \ \theta_{s-2} \leqslant \alpha.$$

Remarque 2. Dans les conditions du théorème si $s = n$ on peut dans l'inégalité (0, 1) remplacer p par 1 pour toutes les fonctions et leurs dérivées jusqu'à l'ordre $(k-1)$ qui se trouvent sous le signe de l'intégrale. Mais ceci n'est pas essentiel. En effet, si une fonction φ est continue ainsi que ses dérivées du premier ordre dans le domaine D et si elle vérifie les conditions.

1) $\int_S^{n-1} \dots \int | \varphi(x' \dots x'_n) |^p dS \leqslant \bar{A}$, où S est une partie unie de la frontière; $p > 1$; $(x' \dots x'_n)$—un point voisin.

2) $\int_D^n \dots \int \sum_{i=1}^{h} \left| \frac{\partial \varphi}{\partial x_i} \right|^p dx_1 \dots dx_n \leqslant \bar{A}$, on a:

$$\int_D^n \dots \int | \varphi(x_1 \dots x_n) |^p \, dx_1 \dots dx_n \leqslant \bar{A} \cdot \bar{B}.$$

Cette inégalité est analogue à celle de Friderichs et peut être appliquée comme cette dernière.

Remarque 3. Supposons qu'on ajoute aux conditions du théorème quelques hypothèses supplémentaires sur les dérivées d'ordre k. Par exemple:

$$1) \int_D^n \dots \int \sum_{\Sigma \alpha_i = k} \left| \frac{\partial^k \varphi}{\partial x_1^{\alpha_1} \dots \partial x_s^{\alpha_s}} \right|^p dx_1 \dots dx_n \leqslant A,$$

ou bien

$$2) \int_D^n \dots \int \sum_{\Sigma \alpha = k} \left| \frac{\partial^k \varphi}{\partial x_1^{\alpha_1} \dots \partial x_s^{\alpha_s}} \right|^p | \xi(\rho_s) |^{p-1} dx_1 \dots dx_n \leqslant A,$$

où les $\xi(\rho_s)$ peuvent, en général, être des fonctions différentes, mais elles doivent vérifier la condition

$$\int_0^a \frac{d\rho}{\rho \, | \, \xi(\rho) \, |} \leqslant A.$$

Dans ces conditions, on voit que $I_{n-s}(\varphi)$ est déjà borné dans le premier cas pour $\tau = \frac{1}{p}$ si $k > \frac{s-1}{p}$; or si $\tau = 1$, il faut prendre $k > \frac{s}{p} - 1$.

Le second cas suppose que le théorème est vrai pour $k = \frac{s}{p}$.

Les résultats indiqués dans ces deux remarques seront démontrés si dans la relation fondamentale, analogue à l'identité (I, 1) et obtenue de l'expression

$$\frac{\partial}{\partial \rho_s}\left\{[\overline{m}k\rho_s^{\overline{m}k+1} - (\overline{m}k+1)\rho_s^{\overline{m}k}\cdot h + h^{\overline{m}k+1}]\omega\right\}$$

(voir l'article de Soboleff (1) pour $\overline{m} = 1$), on considère dans le second membre une somme d'intégrales.

II. Continuité égale moyenne.

Après avoir construit autour du point à coordonnées $\chi_1 \chi_2 \dots x_{s+1} \dots x_n$ un système de coordonnées qui est, en général, tourné d'un certain angle par rapport au système $\rho_s, \Theta_{s-2}, \dots \Theta_1, \psi$ en appliquant l'identité (I, 1) à un point de la multiplicité F et au point correspondant de la multiplicité voisine, on obtient la relation (voir l'article de Soboleff):

$$\varphi(\chi_1 \dots \chi_s x_{s+1} \dots x_n) - \varphi(0, 0 \dots x_{s+1} \dots x_n) = C_1\left\{\int_{v_1 - v_2}^{S}\dots\int \Lambda_1\, dx_1 \dots dx_s + \right.$$

$$\left. + \int_{v_2 - v_1}^{S}\dots\int \Lambda_2\, dx_1 \dots dx_s + \int_{v_1 \cdot v_2}^{S}\dots\int (\Lambda_2 - \Lambda_1)\, dx_1 \dots dx_s \right\} =$$

$$= C_1[I_1 + I_2 + I_3] \qquad \text{(II, 1)}$$

où Λ_1 est l'expression sous le signe d'intégrale dans l'identité (I, 1) et Λ_2 est le résultat qu'on obtient en remplaçant dans Λ_1 les valeurs de $\rho_s, \theta_{s-2}, \dots \theta_1, \psi$ par $\rho_s', \theta'_{s-2}, \dots \theta_1', \psi'$;

$v_1 - v_2$ est la partie du «secteur» S qui n'appartient pas au «secteur» S';

$v_2 - v_1$, la partie du secteur S' qui n'appartient pas au secteur S;

$v_1 \cdot v_2$, la partie commune des deux secteurs.

Pour que tous les résultats de la suite soient vrais nous supposons que l'angle entre les «secteurs atteignants S» d'un point de la multiplicité fondamentale F et du point correspondant d'une multiplicité voisine est d'ordre $N\delta$, où N est une constante absolue.

Il suit de la relation (II, 1) que l'on a:

$$|\varphi(\chi_1 \chi_2 \dots \chi_s x_{s+1} \dots x_n) - \varphi(0_1 0 \dots x_{s-1} x_{s-2} \dots x_n)|^p \leqslant$$

$$\leqslant C_2[|I_1|^p + |I_2|^p + |I_3|^p]. \qquad \text{(II, 2)}$$

L'évaluation pour I_1 prend la forme

$$I_1 \leqslant C_3\left(\int_{\sigma}^{S}\dots\int \sum_{l=0}^{k}\sum_{\Sigma a = l}\left|\frac{\partial^l \varphi}{\partial x_1^{a_1} \dots \partial x_s^{a_s}}\right|^p dx_1 \dots dx_s\right)^{\frac{1}{p}} \begin{cases} \delta^{\frac{q+1}{q}}, & \text{si } \nu > q; \\ \delta^{\frac{q+\varepsilon}{q}}, & \text{si } \nu = q; \\ \delta^{\frac{\nu+1}{q}}, & \text{si } \nu < q. \end{cases} \qquad \text{(II, 3)}$$

1) σ est une couche d'épaisseur $N\delta$ près de la frontière de v_1;

2) $\nu = qk - qs + s - 1$;

3) $q = \frac{p}{p-1} > 1$.

On évalue I_2 d'une manière analogue. (II, 4)

L'évaluation de I_3:

$$I_3 = \int_{v_1-v_2}^{S}\dots\int (\Lambda_2 - \Lambda_1)\, dx_1 \dots dx_s \leqslant \int_{\rho_s<4\delta}\dots\int (|\Lambda_1| + |\Lambda_2|)\, dx_1 \dots dx_s + \\ + \int_{\rho_s\geqslant 4\delta}^{S}\dots\int |\Lambda_2 - \Lambda_1|\, dx_1 \dots dx_s \qquad (II,5)$$

$\rho_s = 4\delta$ est une circonférence décrite autour du point $(0,0\dots 0)$.

On a ensuite

$$\int_{\rho_s<4\delta}^{S}\dots\int \sum_{i=1}^{2} |\Lambda_i|\, dx_1 \dots dx_s \leqslant \\ \leqslant C_s \Big(\int_{\rho_s<4\delta}^{S}\dots\int \sum\sum \Big| \frac{\partial^l \varphi}{\partial x^{\alpha_1} \dots \partial x_s^{\alpha_s}} \Big|\, dx_1 \dots dx_s \Big)^{\frac{1}{p}} \delta^{\frac{\nu+1}{q}}. \qquad (II,6)$$

Pour apprécier l'ordre de grandeur de la seconde composante, on peut profiter de l'évaluation de l'expression $|\Lambda_2 - \Lambda_1|$ donnée par S. L. Soboleff:

$$|\Lambda_2 - \Lambda_1| \leqslant A_1 \delta \rho^{k-s-1} \sum_{0}^{k} \sum_{\Sigma\alpha=l} \Big| \frac{\partial^l \varphi}{\partial x_1^{\alpha_1} \dots \partial x_s^{\alpha_s}} \Big|,$$

d'où

$$\int_{\rho_s\geqslant 4\delta}^{S}\dots\int |\Lambda_2 - \Lambda_1|\, dx_1 \dots dx_s \leqslant \\ \leqslant C_6 \Big(\int_{\rho_s\geqslant 4\delta}^{S}\dots\int \sum_{0}^{k}\sum_{\Sigma\alpha=l} \Big| \frac{\partial^l \varphi}{\partial x_1^{\alpha_1} \dots \partial x_s^{\alpha_s}} \Big|^p dx_1 \dots dx_s \Big)^{\frac{1}{p}} \cdot \delta \Big(\int_{h>\rho_s>4\delta}^{S}\dots\int \rho_s^{\nu-q}\, d\rho_s\, d\gamma \leqslant \\ \leqslant C_7 \Big(\int_{h\geqslant\rho_s\geqslant 4\delta}^{S}\dots\int \sum_{0}^{k}\sum_{\Sigma\alpha=l} \Big| \frac{\partial^l \varphi}{\partial x_1^{\alpha_1} \dots \partial x_s^{\alpha_s}} \Big|^p dx_1 \dots dx_s \Big)^{\frac{1}{p}} \begin{cases} \delta, & \text{si } \nu > q-1, \\ \delta\, |\lg\delta|^{\frac{1}{q}}, & \text{si } \nu = q-1, \\ \delta^{\frac{\nu+1}{q}}, & \text{si } \nu < q-1. \end{cases} \qquad (II,7)$$

En comparant les relations (II, 3) (II, 4) (II, 6) (II, 7) et en tenant compte de l'inégalité (II, 2) on obtient

$$|\varphi(\gamma_1\gamma_2 \dots x_{s+1} \dots x_n) - \varphi(0,0 \dots x_{s+1} \dots x_n)|^p \leqslant \\ \leqslant C_8 \int_{h\geqslant\rho_s}^{S}\dots\int \sum_{0}^{k}\sum_{\Sigma\alpha=l} \Big| \frac{\partial^l \varphi}{\partial x_1^{\alpha_1} \dots \partial x_s^{\alpha_s}} \Big|^p dx \dots dx_s \begin{cases} \delta^p, & \text{si } m > p, \\ \delta\, |\lg\delta|^{p-1}, & \text{si } m = p, \\ \delta^m, & \text{si } m < p. \end{cases} \qquad (II,8)$$

On a remarqué ici que

$$\nu = \frac{pk-s}{p-1} - 1 \quad \text{et} \quad k = \frac{s+m}{p};\ m > 0.$$

En intégrant l'inégalité (II, 8) sur F et en prenant une inégalité qui est évidemment plus forte, on peut écrire

$$\frac{1}{|F|} \int_{F}^{n+s}\dots\int |\varphi(\gamma_1 \dots \gamma_s x_{s+1} \dots x_n) - \varphi(0,0 \dots x_{s+1} \dots x_n)|^p \leqslant \\ \leqslant C_9 \int_{D}^{n}\dots\int \sum_{0}^{k}\sum_{\Sigma\alpha=l} \Big| \frac{\partial^l \varphi}{\partial x_1^{\alpha_1} \dots \partial x_s^{\alpha_s}} \Big|^p dx_1 \dots dx_n \begin{cases} \delta^p, & \text{si } m > p, \\ \delta^p\, |\lg\delta|^{p-1}, & \text{si } m = p, \\ \delta^m, & \text{si } m < p, \end{cases}$$

ce qui démontre le théorème.

La remarque 1 conserve son sens pour une multiplicité voisine. Dans la remarque 2 pour qu'elle subsiste dans le cas de l'égale continuité il faut prendre $p > 1$.

Enfin si dans la remarque 3 on prend dans la condition 1 la valeur de τ égale à $\frac{1}{p}$, on peut écrire:

$$\frac{1}{|F|}\int_F^{n-s}\cdots\int |\varphi(\chi_1\chi_2\ldots\chi_s x_{s+1}\ldots x_n) - \varphi(0,0\ldots x_{s+1}\ldots x_n)|^p\, dx_{s+1}\ldots dx_n \leqslant A \begin{cases} \delta^p, & \text{si } k > \frac{s+p-1}{p}, \\ \delta^p |\lg\delta|^{p-1}, & \text{si } k = \frac{s+p-1}{p}, \\ \delta^{p(k+1)-s}, & \text{si } k < \frac{s+p-1}{p}. \end{cases}$$

III. L'impossibilité de diminuer le nombre des dérivées ou de trouver de meilleurs évaluations.

Cette impossibilité peut être prouvée par des exemples.

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LITTÉRATURE CITÉE

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MATHEMATIK

ÜBER DIE BESTE ANNÄHERUNG ANALYTISCHER FUNKTIONEN

Von N. ACHYESER

(Vorgelegt von S. N. Bernstein, Mitglied der Akademie, am *26. XII. 1937)*

Es sei $\mathfrak{B}_q\,(0 < q < 1)$ die Menge aller analytischen Funktionen $f(z)$, welche:

a) innerhalb der Ellipse $\mathfrak{E}_q$ mit den Brennpunkten ± 1 und der Halbachsensumme $\frac{1}{q}$ regulär sind und der Ungleichheit

$$-1 < \Re f(z) < 1$$

genügen,

b) auf der reellen Achse reell sind.

Ist $E_{n-1}(f)$ der Fehler der besten Annäherung der Funktion $f(x)$ im Intervalle $< -1, 1 >$ durch einen Polynom vom Grade $< n$, so besteht nach einem bekannten Satze von S. Bernstein die Ungleichheit

$$\overline{\lim_{n\to\infty}} \sqrt[n]{E_n(f)} \leqslant q.$$

Im Zusammenhang mit einigen neuen Untersuchungen (1) habe ich eine Möglichkeit für die genaue Bestimmung der Grösse

$$\sup_{f \subset \mathfrak{B}_q} E_{n-1}(f)$$

gefunden.

Ich setze $U(x, y) = \Re f(z)$ und bezeichne durch $E_{n-1}(U; r)$ $(q < r \leqslant 1)$ den Fehler der besten Annäherung in $\mathfrak{E}_r$ der Funktion $U(x, y)$ durch einen harmonischen Polynom vom Grade $< n$. Offenbar ist

$$E_{n-1}(U; 1) = E_{n-1}(f).$$

Wir wollen zeigen, wie der genaue Wert von

$$\sup_{f \subset \mathfrak{B}_1} E_{n-1}(U; r)$$

bestimmt werden kann.

Vermöge der Abbildung $z = \frac{1}{2}\left(w + \frac{1}{w}\right)$ erhalten wir eine Funktion $g(w) = f(z)$, welche eindeutig und regulär im Ringe $q < |w| < \frac{1}{q}$ ist und rein reelle Werte auf $|w| = 1$ hat. Es mögen $\varphi(\theta)$ die Randwerte von $\Re g(w) = U(x, y)$ auf der Kreislinie $w = \frac{1}{q} e^{i\theta}$ oder, was dasselbe ist, auf der Kreislinie $w = q e^{i\theta}$ bedeuten.

In der Laurentschen Entwicklung

$$g(w) = \sum_{-\infty}^{\infty} a_m w^m$$

ist

$$a_m = \frac{1}{2\pi i} \oint \frac{g(w)}{w^{m+1}} dw = \frac{1}{2\pi} \int_0^{2\pi} g\left(\frac{1}{q} e^{i\theta}\right) q^m e^{-im\theta} d\theta =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} g(qe^{i\theta}) q^{-m} e^{-im\theta} d\theta$$

oder

$$a_m = \frac{q^m}{1+q^{2m}} \frac{1}{\pi} \int_0^{2\pi} \varphi(\theta) e^{-im\theta} d\theta.$$

Folglich ist

$$g(w) = \frac{1}{2\pi} \int_0^{2\pi} \left\{1 + 4 \sum_{m=1}^{\infty} \frac{q^m}{1+q^{2m}} \cos m \ (t - \theta + is)\right\} \varphi(\theta) d\theta,$$

wo

$$w = e^{i(t+is)} \qquad \left(q < e^{-s} < \frac{1}{q}\right).$$

Erinnert man an die aus der Theorie der elliptischen Funktionen bekannte Formel

$$1 + 4 \sum_{m=1}^{\infty} \frac{q^m}{1+q^{2m}} \cos m u = \frac{2K}{\pi} dn \frac{Ku}{\pi}$$

und beachtet man das Additionstheorem für dn, so ergibt sich, dass

$$U(x,y) = \Re g(w) = \frac{K}{\pi^2} \int_{-\pi}^{\pi} \varphi(t-u) \frac{\sqrt{1+k^2\rho^2}\, dn \frac{Ku}{\pi}}{1+\rho^2-\rho^2 dn^2 \frac{Ku}{\pi}} du = \int_{-\pi}^{\pi} \Omega(u) \varphi(t-u) du,$$

wo

$$\rho^2 = -sn^2 \frac{iKs}{\pi} > 0$$

ist.

Will man in $\mathfrak{E}_r$ die Funktion $U(x,y)$ am besten durch einen harmonischen Polynom vom Grade $\leqslant n$ approximieren, so muss man offenbar am besten die periodische Funktion $\Re g(e^{-s+it})$ für $e^{-s} = r$ (oder $e^s = r$) durch die Summe

$$\sum_{m=0}^{n-1} (A_m \cos mt + B_m \sin mt)$$

annähern.

Daraus [wenn man beachtet, dass $\Omega(u)$ eine gerade Funktion ist] folgt

$$E_{n-1}(U;r) \leqslant \min_{C_m} 2 \int_0^{\pi} \left| \Omega(u) - \sum_{m=0}^{n-1} C_m \cos mu \right| du =$$

$$= \min_{C_m^*} 2 \int_0^{\pi} \left| \Omega(u) - \sum_{m=0}^{n-1} C_m^* \cos^{2m} \frac{u}{2} \right| du = B_n(q;r). \qquad (1)$$

Um $B_n(q;r)$ zu berechnen, wollen wir zeigen, dass bei beliebigen C_m^* die Differenz

$$\Omega(u) - \sum_{m=0}^{n-1} C_m^* \cos^{2m}\frac{u}{2}$$

in $(0,\pi)$ nicht mehr als n Nullstellen hat. In der Tat sind alle Koeffizienten der Entwicklung von $\Omega(u)$ nach Potenzen von $dn\frac{Ku}{\pi}$ nichtnegativ, und anderseits folgt aus der bekannten Formel

$$dn\frac{Ku}{\pi} = \sqrt{k'}\prod_{m=1}^{\infty}\frac{1+2q^{2m-1}\cos u + q^{4m-2}}{1-2q^{2m-1}\cos u + q^{4m-2}},$$

dass alle Ableitungen von $dn\frac{Ku}{\pi}$ nach der Veränderlichen $\cos^2\frac{u}{2}$ auch nichtnegativ sind.

Dieser Umstand führt zu der Gleichung:

$$B_n(q;r) = \left|\int_{-\pi}^{\pi}\Omega(u)\cdot \operatorname{sgn}\cos nu\cdot du\right| =$$

$$= \frac{1}{2\pi}\left|\Re\int_{-\pi}^{\pi}\left\{1+4\sum_{m=1}^{\infty}\frac{q^m}{1+q^{2m}}\cos m(u+is)\right\}\cdot\operatorname{sgn}\cos nu\cdot du\right| =$$

$$= \frac{4}{\pi}\sum_{m=0}^{\infty}(-1)^m\frac{1}{2m+1}\frac{r^{n(2m+1)}+r^{-n(2m+1)}}{q^{n(2m+1)}+q^{-n(2m+1)}}.$$

Nun nehmen wir die Funktion $f_0(z)$, für welche

$$\varphi_0(\theta) = \operatorname{sgn}\cos n\theta$$

und folglich

$$g_0(w) = \frac{8}{\pi}\sum_{m=0}^{\infty}(-1)^m\frac{1}{2m+1}\frac{\cos n(2m+1)(t+is)}{q^{n(2m+1)}+q^{-n(2m+1)}},$$

sodass

$$U_0(x,y) = \Re g_0(w) = \frac{4}{\pi}\sum_{m=0}^{\infty}(-1)^m\frac{1}{2m+1}\frac{r^{n(2m+1)}+r^{-n(2m+1)}}{q^{n(2m+1)}+q^{-n(2m+1)}}\cos n(2m+1)t$$

ist.

Wir sehen, dass an den Stellen:

$$(w) = r,$$

$$\arg w = t = \frac{2\lambda+1}{2n}\pi \qquad (\lambda = -n, -(n-1), \ldots, n-1),$$

die Funktion $\Re g_0(w)$ mit abwechselnden Vorzeichen den Wert $B_n(q;r)$ annimmt. Daraus folgt nach (1), dass

$$E_{n-1}(U_0;r) = B_n(q;r),$$
$$\sup_{f\subset\mathfrak{B}_q} E_{n-1}(U;r) = B_n(q;r).$$

Die extremale Funktion $f_0(z)$ ist dadurch vollständig charakterisiert, dass am Rande von $\mathfrak{E}_r$ ihr reeller Teil abwechselnd ± 1 ist auf gewissen Bogen, welche mit Hilfe eines Systems von Hyperbeln mit den Brennpunkten ± 1 bestimmt werden können.

Man kann die Funktion $f_0(z)$ auch als das Produkt aus $\frac{i}{\pi}$ und dem

Logarithmus einer elliptischen Funktion darstellen, aber hierauf wollen wir nicht eingehen.

Für $r = 1$ wird

$$\sup_{f \subset \mathfrak{B}_q} E_{n-1}(f) = \frac{8}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \frac{q^{n(2m+1)}}{1+q^{2n(2m+1)}} = B_n(q).$$

Man sieht leicht, dass

$$B_n(q) = \frac{8}{\pi} \sum_{k=0}^{\infty} (-1)^k \operatorname{arctg} q^{n(2k+1)}$$

und also

$$\operatorname{arctg} q^n - \operatorname{arctg} q^{3n} < \frac{\pi}{8} B_n(q) < \operatorname{arctg} q^n.$$

Wir bemerken noch, dass unser Ergebnis auch die genaue obere Schranke gibt für die beste Annäherung durch eine trigonometrische Summe von der Ordnung $< n$ einer reellen periodischen Funktion $F(e^{i\vartheta})$, falls $F(e^{i\vartheta - \sigma})$ regulär in dem Streifen $-h < \sigma < h$ ist und falls in diesem Streifen $-1 < \Re F < 1$ ist.

Einfache Überlegungen, von welchen wir hier Gebrauch gemacht haben, erlauben auch, auf andere Fragen eine Antwort zu geben. Wir geben die folgende: die Anfangstemperatur im Ringe

$$f(\varphi) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\varphi + b_k \sin k\varphi)$$

möge der Ungleichheit $-1 \leqslant f(\varphi) \leqslant 1$ genügen; zur Zeit t hat die Temperatur den Wert

$$f(\varphi; t) = e^{-bt} \left\{ \frac{a_0}{2} + \sum_{k=1}^{\infty} e^{-k^2 a^2 t} (a_k \cos k\varphi + b_k \sin k\varphi) \right\};$$

bedeutet $E_{n-1}(f; t)$ den Fehler der besten Annäherung durch eine trigonometrische Summe von der Ordnung $< n$ der Temperatur zur Zeit t, so soll man die genaue obere Grenze für $E_{n-1}(f; t)$ finden und auch die Anfangstemperatur, welcher sie entspricht. Wir haben:

$$f(\varphi; t) = \frac{1}{2\pi} e^{-bt} \int_{-\pi}^{\pi} \left\{ 1 + 2 \sum_{k=1}^{\infty} e^{-k^2 a^2 t} \cos k\psi \right\} f(\varphi + \psi)\, d\psi =$$

$$= \frac{1}{2\pi} e^{-bt} \int_{0}^{\pi} \Theta_1\left(\frac{K\psi}{\pi}\right) \{ f(\varphi + \psi) + f(\varphi - \psi) \}\, d\psi,$$

wo für die elliptische ϑ-Funktion $q = e^{-a^2 t}$ ist.

Benutzt man die Entwicklung der Funktion Θ_1 in ein unendliches Produkt, so ergibt sich, dass alle Ableitungen von Θ_1 nach der Veränderlichen $\cos^2 \frac{\psi}{2}$ nichtnegativ sind. Daraus folgt, dass die gesuchte obere Grenze gleich

$$\frac{4}{\pi} e^{-bt} \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} e^{-(2m+1)^2 n^2 a^2 t}$$

ist und für jedes t nur dann erreicht wird, wenn

$$f(\varphi) = \operatorname{sgn} \cos n\varphi.$$

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MATHÉMATIQUES

SUR QUELQUES POINTS DE LA THÉORIE DE LA MEILLEURE APPROXIMATION DES FONCTIONS PÉRIODIQUES

Par M. KREIN

(Présenté par S. N. Bernstein, de l'Académie, le 26. XII. 1937)

La présente note se rattache étroitement à quelques recherches récentes de J. Favard (1), N. Achyeser (2,4) et de l'auteur.

Soit

$$G(\varphi) \sim \sum_{-\infty}^{\infty} \mu_k e^{ki\varphi} \quad (\mu_{-k} = \bar{\mu}_k; \quad k = 0, 1, 2, \ldots)$$

une fonction réelle continue de période 2π. Posons

$$H_n(\varphi) = G(\varphi) - G\left(\varphi + \frac{\pi}{n}\right) + G\left(\varphi + \frac{2\pi}{n}\right) - \ldots - G\left(\varphi + \frac{(2n-1)\pi}{n}\right).$$

Comme $H_n\left(\varphi + \frac{\pi}{n}\right) = -H_n(\varphi)$, il existe des points dans lesquels la fonction $H_n(\varphi)$ s'annule.

Déterminons les valeurs $\xi_k(\alpha)$ $(k=0, 1, \ldots n-1)$ et $\eta_k(\alpha)$ $(k=1, \ldots n-1)$ de manière que la fonction

$$R(\varphi) = G(\varphi) - \sum_{0}^{n-1} (\xi_k(\alpha) \cos k\varphi + \eta_k(\alpha) \sin k\varphi)$$

s'annule aux points α, $\alpha + \frac{\pi}{n}$, ... , $\alpha + \frac{(2n-2)\pi}{n}$. Cette fonction aura le point $\alpha + \frac{(2n-1)\pi}{n}$ pour zéro dans le cas et dans ce cas seulement où

$$H_n(\alpha) = 0.$$

En effet

$$R\left(\alpha + \frac{(2n-1)\pi}{n}\right) = -\sum_{k=0}^{2n-1} (-1)^k R\left(\alpha + \frac{k\pi}{n}\right) = -H_n(\alpha).$$

Convenons de dire que la fonction $G(\varphi)$ possède la propriété (A_n), pour tout α tel que $H_n(\alpha) = 0$ la fonction $R(\varphi)$ change de signe en même temps que $\sin n(\varphi - \alpha)$.

Lemme 1. *Si la fonction $G(\varphi)$ possède la propriété (A_n), on a*

$$\min_{\xi, \eta} \int_0^{2\pi} \left| G(\varphi) - \sum_{0}^{n-1} (\xi_k \cos k\varphi + \eta_k \sin k\varphi) \right| d\varphi = \max F_n(\varphi),$$

où

$$F_n(\varphi) = \sum_{k=1}^{2n} (-1)^k \int_{\varphi+\frac{(k-1)\pi}{n}}^{\varphi+\frac{k\pi}{n}} G(\varphi)\, d\varphi \sim \frac{4}{i} \sum_{k=-\infty}^{\infty} \frac{\mu_{(2k+1)n} e^{(2k+1)ni}}{2k+1}.$$

Tous les extrêma de la fonction $F_n(\varphi)$ ont la même valeur absolue et sont atteints aux points α, où $H_n(\alpha) = 0$.

Démonstration. Soit α un zéro quelconque de la fonction $H_n(\varphi)$. Quels que soient les ξ_k et η_k on a

$$\int_0^{2\pi} \left| G(\varphi) - \sum_0^{n-1} (\xi_k \cos k\varphi + \eta_k \sin k\varphi) \right| d\varphi \geqslant$$

$$\geqslant \left| \int_0^{2\pi} \left\{ G(\varphi) - \sum_0^{n-1} (\xi_k \cos k\varphi + \eta_k \sin k\varphi) \right\} \operatorname{sgn} \sin n(\varphi - \alpha)\, d\varphi \right| =$$

$$= \left| \int_0^{2\pi} G(\varphi) \operatorname{sgn} \sin n(\varphi - \alpha)\, d\varphi \right| = |F_n(\alpha)|.$$

D'autre part le signe $=$ dans la première relation est réalisé pour $\xi_k = \xi_k(\alpha)$ $(k = 0, 1, \ldots n-1)$ et $\eta_k = \eta_k(\alpha)$ $(k = 1, 2, \ldots n-1)$ ce qui achève la démonstration.

Désignons par $\mathfrak{M}$ l'ensemble de toutes les fonctions $f(\varphi)$ qui admettent la représentation

$$f(\varphi) = \int_0^{2\pi} G(\varphi - \psi)\, h(\psi)\, d\psi,$$

où $h(\psi)$ est une fonction mesurable, ayant, en général, des valeurs complexes et satisfaisant à l'inégalité:

$$|h(\psi)| \leqslant 1.$$

Désignons, comme d'ordinaire, par $E_{n-1}(f)$ la meilleure approximation de $f(\varphi)$ par un polynôme trigonométrique de degré $n-1$ au plus.

Lemme 2. *Si la fonction $G(\varphi)$ possède la propriété (A_n) on a:*

$$\sup_{f \subset \mathfrak{M}} E_{n-1}(f) = \max F_n(\varphi).$$

Le supremum est atteint pour la fonction $f_0(\varphi) = F_n(\varphi)$ correspondant à la fonction $h(\varphi) = -\operatorname{sgn} \sin n\varphi$.

Démonstration. En effet la fonction atteint son maximum absolu en changeant de signe alternativement en $2n$ points α, $\alpha + \frac{\pi}{n}$, ... , $\alpha + \frac{(2n-1)\pi}{n}$; par conséquent, d'après le théorème de Tchebycheff

$$E_{n-1}(f_0) = \max f_0(\varphi) = \max F_n(\varphi).$$

D'autre part, si

$$f(\varphi) = a_0 + \sum_{k=1}^{\infty} (a_k \cos k\varphi + b_k \sin k\varphi) \tag{2}$$

est une fonction arbitraire de $\mathfrak{M}$ et

$$2\mu_k a'_k = \xi_k(\alpha)\, a_k - \eta_k(\alpha)\, b_k,$$

$$\mu_0 a_0' = a_0 \xi_0(\alpha), \qquad (k = 1, 2, \ldots, n-1),$$
$$2\mu_k b_k' = \eta_k(\alpha) a_k + \xi_k(\alpha) b_k,$$

on a la relation

$$\left| f(\varphi) - \sum_0^{n-1} (a_k' \cos k\varphi + b_k' \sin k\varphi) \right| =$$
$$= \left| \int_0^{2\pi} \left\{ G(\psi) - \sum_0^{n-1} (\xi_k(\alpha) \cos k\psi + \eta_k(\alpha) \sin k\psi) \right\} h(\varphi - \psi)\, d\psi \right| \leqslant$$
$$\leqslant \int_0^{2\pi} \left| G(\psi) - \sum_0^{n-1} (\xi_k(\alpha) \cos k\psi + \eta_k(\alpha) \sin k\psi) \right| d\psi = \max F_n(\varphi),$$

d'où $E_{n-1}(f) \leqslant \max F_n(\varphi)$.

Lemme 3. *Soit $P(D) = D^2 + pD + q$ un trinôme réel quadratique tel que les parties imaginaires de ses racines ne surpassent pas un nombre $\mu > 0$. Si une fonction $f(x)$ dont les deux premières dérivées sont continues, possède $n+2$ zéros $\alpha_0 < \alpha_1 < \ldots < \alpha_n < \alpha_{n+1} (n \geqslant 1)$ tels que*

$$|\alpha_i - \alpha_{i+1}| \leqslant \frac{\pi}{\mu} \quad (i = 0, 1 \ldots n),$$

la fonction $y(x) = P(D) f(x)$ a au moins n nœuds à l'intérieur de l'intervalle (α_0, α_{n+1}) ou bien elle s'annule identiquement sur une partie de cet intervalle.

Ce lemme forme un cas particulier d'une proposition générale, concernant des opérateurs différentiels de Sturm-Liouville. La démonstration de cette dernière proposition sera donnée dans un autre Recueil. On déduit sans peine du lemme 3 le

Lemme 4. *Soit*

$$P(D) = D^p + A_1 D^{p-1} + \ldots + A_p$$

un polynôme réel quelconque qui possède $k \left(0 \leqslant k \leqslant \left[\frac{p}{2} \right] \right)$ couples de racines complexes et soit $\mu > 0$ un nombre non inférieur à chacune des parties imaginaires de ces racines.

Si la fonction périodique $f(\varphi)$ de période 2π dont les p premières dérivées sont continues, possède $2n$ zéros $\alpha_1 < \alpha_2 < \ldots < \alpha_{2n} (0 \leqslant \alpha_1,\ \alpha_{2n} < 2\pi)$ tels que

$$|\alpha_i - \alpha_{i+1}| < \delta \quad (i = 1, 2, \ldots 2n;\ \alpha_{2n+1} = \alpha_1 + 2\pi),$$

*où $\delta = \frac{\pi}{\mu 3^{k-1}}$ *, la fonction $g(\varphi) = P(D) f(\varphi)$ a au moins $2n$ nœuds dans l'intervalle $(0, 2\pi)$ ou bien elle s'annule identiquement sur une partie de cet intervalle.*

Théorème 1. *Soit $\mathfrak{M}$ l'ensemble de toutes les fonctions $f(\varphi)$ périodiques de période 2π, admettant p dérivées, et telles que*

$$|f^{(p)}(\varphi) + A_1 f^{(p-1)}(\varphi) + \ldots + A_p f(\varphi)| \leqslant 1.$$

*Il existe un nombre N tel que pour $n > N$ ** on a:*

$$\sup_{f \subset \mathfrak{M}} E_{n-1}(f) = \frac{2}{\pi} \max \left| \sum_{k=-\infty}^{\infty} \frac{e^{(2k+1)ni\varphi}}{(2k+1) P[(2k+1) ni]} \right|.$$

---

* Il est probable que le facteur 3^{k-1} dans l'expression de δ peut être supprimé.
** Et pour toutes les valeurs de n si les racines de $P(D)$ sont toutes réelles.

Démonstration. Pour simplifier nos considérations, supposons que le polynôme $P(D) = D^p + A_1 D^{p-1} + \ldots + A_p$ ne s'annule en aucun des points ki $(k = 0, 1, 2, \ldots)$.

Si l'on pose

$$G(\varphi) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \frac{e^{ki\varphi}}{P(ki)}$$

les fonctions $f(\varphi) \subset \mathfrak{M}$ et $h(\varphi)$ se trouvent liées par la relation (1). Pour démontrer le théorème il ne reste donc qu'à montrer que pour n suffisamment grand la différence

$$R(\varphi) = G(\varphi) - \sum_{k=0}^{n-1} (\xi_k(\alpha) \cos k\varphi + \eta_k(\alpha) \sin k\varphi)$$

change de signe en même temps que $\sin n(\varphi - \alpha)$. Admettant que pour un $n > 3^{k-1}\mu$ il n'en est pas ainsi, on pourrait affirmer que pour certains ζ_k $(k = 0, 1, \ldots n-1)$ qui ne s'annulent pas simultanément et sont assez proches des valeurs correspondantes $\zeta_k(\alpha)$ $[\zeta_0(\alpha) = \xi_0(\alpha),\ 2\zeta_k(\alpha) = \xi_k(\alpha) - i\eta_k(\alpha),\ (k = 1, 2, \ldots n-1)]$ la fonction

$$R_1(\varphi) = G(\varphi) - \sum_{-n+1}^{n-1} \zeta_k e^{ki\varphi} (\zeta_{k_i} = \overline{\zeta}_{-k};\quad k = 1, 2. \ldots n-1)$$

a au moins $2n+2$ zéros dans l'intervalle $< 0, 2\pi)$ et la distance entre les zéros voisins de cette fonction est inférieure à $\delta = \frac{\pi}{3^{k-1}\mu}$.

Soit

$$P(D) = Q(D)(D^2 + pD + q).$$

D'après le lemme 4 la fonction $Q(D) R_1(\varphi)$ a $2n+2$ zéros au moins dans l'intervalle $< 0, 2\pi)$ et la distance entre deux zéros voisins est $\leqslant \frac{\pi}{\mu}$. Donc d'après le lemme 3 la fonction

$$P(D) R_1(\varphi) = (D^2 + pD + q) Q(D) R_1(\varphi) = -\sum_{-n+1}^{n-1} P(ki) \zeta_k e^{ki\varphi} [P(D) G \equiv 0]$$

a à l'intérieur de l'intervalle $< 0, 2\pi > 2n$ zéros ce qui est impossible. Le théorème est donc démontré.

Remarquons que si l'opérateur $P(D)$ ne contient que des puissances impaires de D on a

$$I_n = \frac{2}{\pi} \left| \sum_{k=-\infty}^{\infty} \frac{1}{(2k+1) P[(2k+1) ni]} \right| \qquad (n > N)$$

et s'il ne contient que des puissances paires de D on a

$$I_n = \frac{2}{\pi} \left| \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(2k+1) P[(2k+1) ni]} \right| \qquad (n > N).$$

Pour le cas particulier $P(D) = D^p$ le théorème 1 a été établi pour la première fois par Favard (3) et un peu plus tard et indépendamment

par Achyeser et l'auteur. Tout récemment Achyeser a démontré le cas particulier du théorème, où *

$$P(D) = \prod_{k=1}^{m} (D^2 + \alpha_k) \quad \text{ou} \quad D\prod_{k=1}^{m} (D^2 + \alpha_k),$$

où α_k $(k = 1, 2, \ldots m)$ sont des nombres réels quelconques.

Faisons correspondre à toute fonction $f(\varphi)$, admettant le développement (2) la fonction

$$f^*(\varphi) = \sum_{1}^{\infty} (b_k \cos k\varphi - a_k \sin k\varphi).$$

On démontre d'une manière analogue le

Théorème 2. *Soit $\mathfrak{M}$ le même ensemble que dans le théorème 1. Il existe un nombre N tel que pour $n > N$, on a*

$$\sup_{f \subset \mathfrak{M}} E_{n-1}(f^*) = \frac{2}{\pi} \max \left| \sum_{k=-\infty}^{\infty} \frac{e^{(2k+1)ni\varphi}}{|2k+1|\, P[(2k+1)ni\varphi]} \right|.$$

Soit maintenant $f(r, \varphi)$ une fonction harmonique à l'intérieur du cercle $r < 1$. Désignons par $E_{n-1}[f; \rho]$ $(\rho < 1)$ sa meilleure approximation par un polynôme harmonique de degré $n-1$ au plus dans le cercle $r \leqslant \rho$. Il est intéressant d'indiquer que les méthodes développées ci-dessus permettent de résoudre la question sur la valeur exacte de l'expression

$$\sup_{f \subset \mathfrak{K}} E_{n-1}[f; \rho], \tag{3}$$

où $\mathfrak{K}$ désigne l'ensemble de toutes les fonctions $f(r, \varphi)$ harmoniques dans le cercle $r < 1$ et satisfaisant dans ce cercle l'inégalité

$$|f(r, \varphi)| \leqslant 1 \quad (0 \leqslant r \leqslant 1,\ 0 \leqslant \varphi \leqslant 2\pi),$$

ou à une inégalité plus générale (et dans ce cas n doit être suffisamment grand)

$$\left| \frac{\partial^p}{\partial r^p} f(r, \varphi) + A_1 \frac{\partial^{p-1}}{\partial r^{p-1}} f(r, \varphi) + \ldots + A_p f(r, \varphi) \right| \leqslant 1 \ (0 \leqslant r < 1;\ 0 \leqslant \varphi \leqslant 2\pi).$$

Dans le premier cas l'expression (3) est égale à $\frac{4}{\pi} \operatorname{arctg} r^n$. Dans les deux cas le supremum est atteint par la fonction

$$f(r, \varphi) = \frac{2}{\pi} \Im \lg \frac{1 + z^n}{1 - z^n}.$$

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* Et dans ce cas, comme l'a montré Achyeser [4], on peut poser $N = \mu$.

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MATHÉMATIQUES

SUR LES COURBES SITUÉES SUR UNE SURFACE À COURBURE NORMALE CONSTANTE

Par B. D. KAMINSKY

(Présenté par S. N. Bernstein, de l'Académie, le 26. XII. 1937)

Nous supposerons que les surfaces considérées sont regulières, que les fonctions qu'on va rencontrer dans cet article sont finies et continues de même que toutes leurs dérivées jusqu'au troisième ordre.

Généralisons la conception de ligne asymptotique dont la courbure normale en chaque point est la même et nulle. A savoir, considérons les courbes sur la surface dont la courbure normale est la même en tous les points et égale à $\frac{1}{R}$. De pareilles courbes seront nommées courbes à courbure normale constante.

L'équation différentielle des courbes pareilles sera

$$(RL - E)\,du^2 + 2(RM - F)\,du\,dv + (RN - G)\,dv^2 = 0, \qquad (1)$$

où E, F, G sont les coefficients de Gauss de la première forme quadratique et L, M, N,—ceux de la deuxième forme quadratique. Par conséquent, par chaque point donné de la surface passent deux courbes á courbure normale constante $\frac{1}{R}$. Ce n'est que dans le cas où la surface est l'enveloppe d'une famille de sphères de rayon constant dont les centres se trouvent sur une certaine courbe, qu'il n'existe qu'une seule ligne à courbure normale constante $\frac{1}{R}$ qui passe par le point donné.

Pour les surfaces de révolution, le problème de la recherche des courbes à courbure normale constante se réduit aux quadratures.

Surtout il est facile d'obtenir les équations de pareilles courbes pour un cylindre droit circulaire et pour un cône circulaire droit.

L'angle Θ entre deux lignes ayant la même courbure normale $\frac{1}{R}$ se calcule d'après la formule:

$$\operatorname{tg}\Theta = \frac{\sqrt{-(R^2K - 2RH + 1)}}{RH - 1}, \qquad (2)$$

où K est la courbure totale et H—la courbure moyenne au point considéré.

2. Si l'on suppose que les courbes à courbure normale constante $\frac{1}{R}$

sont paramétriques, les coefficients de Gauss doivent satisfaire aux conditions:

$$\frac{L}{E} = \frac{N}{G} = \frac{1}{R}, \quad \frac{M}{F} \neq \frac{1}{R}. \tag{3}$$

En ce cas la torsion ordinaire des lignes paramétriques peut être calculée d'après les formules:

$$\frac{1}{\tau_{(u)}} = \frac{\omega \cdot \left[\frac{G}{R} \cdot \left\{\begin{matrix}12\\1\end{matrix}\right\} \cdot \left\{\begin{matrix}12\\1\end{matrix}\right\} + \left\{\begin{matrix}22\\1\end{matrix}\right\}_v + \frac{G^2(F - RM)}{R^2\omega^2} - \left\{\begin{matrix}22\\1\end{matrix}\right\}^2 M - \frac{G_v}{R}\left\{\begin{matrix}22\\1\end{matrix}\right\}\right]}{\omega^2\left\{\begin{matrix}22\\1\end{matrix}\right\}^2 + \frac{G^2}{R^2}}$$

$$\frac{1}{\tau_{(v)}} = \frac{\omega \cdot \left[\left\{\begin{matrix}11\\2\end{matrix}\right\}^2 M + \frac{E_u}{R}\left\{\begin{matrix}11\\2\end{matrix}\right\} - \frac{E}{R}\left\{\begin{matrix}11\\2\end{matrix}\right\} \cdot \left\{\begin{matrix}12\\2\end{matrix}\right\} + \frac{E^2(F - RM)}{R^2\omega^2} + \left\{\begin{matrix}11\\2\end{matrix}\right\}_u\right]}{\omega^2 \cdot \left\{\begin{matrix}11\\2\end{matrix}\right\}^2 + \frac{E^2}{R^2}}.$$

où $\omega = \sqrt{\cos - F^2}$, et $\left\{\begin{matrix}12\\1\end{matrix}\right\}$, ... sont les symboles bien connus de Christophel à trois indices. On calcule ensuite la torsion géodesique des lignes paramétriques. Il se trouve que

$$a_{(u)} = \frac{F - RM}{\omega}, \quad a_{(v)} = -\frac{F - RM}{\omega}. \tag{4}$$

Il en suit que les courbes à courbure normale constante ont des torsions géodésiques égales en valeur absolue mais de signes contraires. On connaissait cette proposition précédemment pour les courbes orthogonales.

Ensuite nous démontrons la proposition inverse: pour deux courbes qui se coupent les torsions géodésiques au point de leur intersection sont égales en valeur absolue et de signes contraires, si ces courbes sont orthogonales ou bien si en ce point leur courbures normales sont égales.

Comme pour les lignes asymptotiques la torsion géodesique coïncide avec leur torsion ordinaire, on déduit des résultats et des formules précédentes le théorème de Beltrami-Enneper sur les lignes asymptotiques.

Ensuite, en prennant la longueur de l'arc d'une courbe à courbure normale constante $\frac{1}{R}$ comme paramètre, on calcule la projection du vecteur tangent à l'image sphèrique d'une pareille courbe sur le vecteur tangent de cette dernière. Il se trouve que cette projection est constante et égale à $-\frac{1}{R}$ pour tous les points de la courbe. De là comme conséquence on obtient l'orthogonalité de la ligne asymptotique à son image sphérique.

Supposons que $\bar{\xi}$ désigne la normale à la surface au point donné et que cette surface est rapportée aux courbes à courbure normale constante $\frac{1}{R}$. Alors on peut obtenir les formules

$$\left.\begin{aligned} \bar{x}_u &= -\frac{\lambda^2 R}{\lambda^2 + R^2}\bar{\xi}_u + \frac{\lambda R^2}{\lambda^2 + R^2}[\bar{\xi} \times \bar{\xi}_u], \\ \bar{x}_v &= -\frac{\lambda^2 R}{\lambda^2 + R^2}\bar{\xi}_v - \frac{\lambda R^2}{\lambda^2 + R^2}[\bar{\xi} \times \bar{\xi}_v]. \end{aligned}\right\} \tag{5}$$

Cela va sans dire que le vecteur $\bar{\xi}$ et la fonction $\lambda(u, v)$ doivent satisfaire à la condition de compatibilité des équations (5).

En posant dans les formules (5) $\frac{1}{R} = 0$ on obtient les formules de Lelièvre pour les lignes asymptotiques.

3. Dans la démonstration de ces dernières formules on doit considérer le vecteur

$$\bar{y} = \bar{x} + R\bar{\xi},$$

c'est-à-dire, l'équation vectoriale de la surface parallèle à la surface donnée et se trouvant à une distance égale à R de cette surface. Il se trouve qu'aux courbes à courbure normale constante correspondent sur la surface $\bar{Y}$ des courbes à courbure normale constante, mais égale à $-\frac{1}{R}$. D'ailleurs, ces dernières forment un angle qui est égal à l'angle des courbes initiales de la surface $\bar{X}$. Aux lignes de courbure de la surface $\bar{X}$ correspondent les lignes de courbure de la surface $\bar{Y}$ et d'ailleurs les equations différentielles de toutes ces courbes sont

$$Edu^2 - Gdv^2 = 0.$$

On démontre ensuite que les courbes à courbure normale constante situées sur la surface X sont orthogonales aux courbes à courbure normale constante $-\frac{1}{R}$ qui leur correspondent sur la surface parallèle Z.

4. Enfin on considère les surfaces ayant un réseau orthogonal de courbes à courbure normale constante $\frac{1}{R}$. De telles surfaces ont la courbure normale constante H égale à $\frac{1}{R}$. Les coefficients de Gauss d'une pareille surface rapportée aux courbes mentionnées prennent la forme:

I) $E = \lambda(u, v) f_1^2(u), \quad F = 0, \quad G = \lambda(u, v) f_2^2(v),$

II) $L = \frac{\lambda(u, v) f_1^2(u)}{R}, \quad M = f_1(u) f_2(v), \quad N = \frac{\lambda(u, v) f_2^2(v)}{R}.$

Ici les fonctions $\lambda(u, v)$, $f_1(u)$ et $f_2(v)$ ne doivent satisfaire qu'à la formule de Gauss, liant la courbure aux coefficients de Gauss de la première forme quadratique, et $\lambda(u, v)$ est le rayon de la courbure géodésique des lignes paramétriques. Il suit des formules pour E et G que sur la surface à courbure moyenne constante $H = \frac{1}{R}$. Les courbes à courbure normale constante $\frac{1}{R}$ forment un réseau isothermique.

Si λ ne dépend que d'une seule variable, par exemple v, la surface présente un cas particulier des surfaces de Liouville. Dans ce cas:

$$E = \lambda(v) f_1^2(u), \quad F = 0, \quad G = \frac{R^2 \lambda(v) \lambda'^2(v)}{C - 4R^2\lambda(v) - 4\lambda^3(v)},$$

$$M = \frac{R f_1(u) \lambda'(v)}{\sqrt{C - 4R^2\lambda(v) - 4\lambda^3(v)}}, \quad L = \frac{\lambda(v) f_1^2(u)}{R}, \quad N = \frac{R\lambda(v)\lambda'(v)}{C - 4R^2\lambda(v) - 4\lambda^3(v)},$$

où $f_1(u)$ et $\lambda(v)$ sont des fonctions arbitraires.

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MATHÉMATIQUES

SUR LES TRANSFORMATIONS INTÉGRALES DE L'ESPACE L

J. SIRVINT

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Une transformation linéaire d'un espace normé peut ne pas être totalement continue, tandis que l'une de ses puissances—et, par conséquent, toutes les suivantes—l'est. Cette circonstance a une grande importance dans la théorie abstraite de Fredholm (1).

Considérons d'abord la classe des transformations linéaires $U(x)$ de l'espace L*:

$$U(x) = \int_0^1 K(s,t)\,x(t)\,dt, \quad \text{où} \quad \operatorname*{vrai\,max}_{0\leqslant s,\ t\leqslant 1} |K(s,t)| = K_0 < \infty. \tag{I}$$

Voici les deux propriétés de cette classe qui vont nous intéresser:

1) parmis les opérations (1) il y en a des opérations non totalement continues,

2) le carré de toute opération (1) est totalement continu.

Pour démontrer 1) introduisons une suite $\{y_n(s)\}$, $n = 1, 2, \ldots$ de fonctions définies dans $<0, 1>$ comme il suit:

$$y_n(s) = \left\{ \begin{array}{l} 1 \text{ quand } \dfrac{2k}{2^n} < s \leqslant \dfrac{2k+1}{2^n} \\ -1 \text{ quand } \dfrac{2k-1}{2^n} < s \leqslant \dfrac{2k}{2^n} \end{array} \right\} \quad (k = 0, 1, \ldots, 2^{n-1}). \tag{1}$$

La suite $\{y_n\}$ des éléments de L ainsi définie n'est pas compacte puisque $\|y_n - y_m\| = 1$ $(n \neq m)$. Déterminons maintenant une suite d'intervalles qui n'empiètent pas $\{d_n\}$, $n = 1, 2, \ldots$ avec $\sum_{n=0}^{\infty} d_n = (0,1)$.

Posons

$$K(s,t) = y_n(s) \text{ quand } t \in d_n. \tag{2}$$

L'opération $U(x)$ avec le noyau (2) appartient à la classe (1). Elle n'est pas totalement continue. En effet, soit, pour chaque d_n,

$$x_n(t) = \left\{ \begin{array}{ll} \dfrac{1}{|d_n|} & \text{quand } t \in d_n, \\ 0 & \text{en dehors de } d_n. \end{array} \right.$$

* Pour la notation, consulter le livre cité sous (1). D'ailleurs, e étant un ensemble linéaire de points et E un ensemble plan, nous allons désigner $m_1 e$ par $|e|$ et $m_2 E$ par $\|E\|$.

La suite $\{x_n\}$ est bornée puisque $\|x_n\| = 1$. On a $U(x_n) = y_n$, et la proposition 1) se trouve démontrée.

Quant à la proposition 2), on peut la démontrer directement, en s'appuyant sur un critère de continuité totale dans L donné par Gelfand (2). Nous préférons cependant suivre une autre voie.

Lemme. *Soit $K(s,t)$ une fonction mesurable dans $0 \leqslant s, t \leqslant 1$, et soit u* vrai max $|K(s,t)| = K_0 < \infty$. *Il existe alors, quel que soit $\varepsilon > 0$, une fonction mesurable $K_\varepsilon(s,t)$, continue par rapport à s, t étant fixé dans un ensemble de mesure 1. On a* vrai max $|K_\varepsilon(s,t)| \leqslant K_0$, *et si l'on désigne par $\varphi(s,t)$ la fonction caractéristique de l'ensemble sur lequel $K(s,t) \neq K_\varepsilon(s,t)$, on a:*

$$\operatorname*{vrai\,max}_{0 \leqslant t \leqslant 1} \int_0^1 \varphi(s,t)\, ds \leqslant \varepsilon.$$

Démonstration. Construisons d'abord une suite décroissante d'ensembles E_n de points du carré $0 \leqslant s, t \leqslant 1$ avec $\|E_n\| = \frac{1}{2^n}$ et une suite de fonctions $K_{\varepsilon,n}(s,t)$ définies respectivement sur E_n, et continues sur ces ensembles.

Désignons par E_0 le carré $0 \leqslant s, t \leqslant 1$. Supposons que les ensembles $E_0, E_1, \ldots, E_n$ sont construits et déterminons la fonction $K_{\varepsilon,n}(s,t)$ et l'ensemble E_{n+1}.

$\|E_n\|$ étant égal à $\frac{1}{2^n}$, il existe sur E_n une fonction continue, dont la valeur absolue ne surpasse pas K_0 et qui est identique à $K(s,t)$ sur un sous-ensemble de E_n de mesure superficielle supérieure à $\frac{1}{2^n}\left(1 - \frac{\varepsilon}{2}\right)$. Cette fonction sera désignée par $K_{\varepsilon,n}(s,t)$ et la fonction caractéristique de l'ensemble, où $K(s,t) \neq K_{\varepsilon,n}(s,t)$, sera désignée par $\varphi_n(s,t)$.

Soit

$$e_n^* = \mathop{\mathcal{E}}_{t}\left[\int_0^1 \varphi_n(s,t)\, ds > \varepsilon\right].$$

Puisque $|e_n^*| < \frac{1}{2^{n+1}}$, il existe un ensemble e_n de points t tel que

$$e_n^* \subset e_n \quad \text{et} \quad |e_n| = \frac{1}{2^{n+1}}.$$

Posons

$$E_{n+1} = \mathop{\mathcal{E}}_{(s,t)}(t \in e_n).$$

Donc, E_{n+1} est un ensemble de segments, parallèles à l'axe des s.

La suite d'ensembles E_n et de fonctions $K_{\varepsilon,n}(s,t)$ étant construite, posons:

$$K_\varepsilon(s,t) = \begin{cases} K_{\varepsilon,n}(s,t) & \sup E_n - E_{n+1}, \\ \text{arbitraire} & \sup \prod_{n=1}^{\infty} E_n. \end{cases}$$

La fonction $K_\varepsilon(s,t)$ ainsi définie vérifie toutes les conditions du lemme.

Remarquons que moyennant la méthode développée ci-dessus on peut démontrer un théorème semblable au théorème connu de M. Lusin sur la «propriété C», à savoir:

«Si $K(s,t)$ est une fonction mesurable dans $0 \leqslant s, t \leqslant 1$, elle est, quel que soit $\varepsilon > 0$, continue par rapport à s dans un ensemble mesu-

rable E tel que l'ensemble des lignes $t = \text{const}$, dont l'intersection avec E a une mesure linéaire inférieure à $1 - \varepsilon$, est de mesure nulle».

Passons maintenant à la démonstration de la proposition 2).

Construisons pour l'opération $U(x)$ (I) une opération $U_\varepsilon(x)$ avec le noyau $K_\varepsilon(s, t)$ défini dans le lemme.

Nous avons alors $\| U - U_\varepsilon \| \leqslant 2K_0\varepsilon$ et, par conséquent,

$$\| U^2 - U_\varepsilon^2 \| \leqslant 4K_0^2\varepsilon. \tag{3}$$

L'opération $U_\varepsilon(x)$ peut être considérée comme une opération linéaire transformant L en C. D'autre part, quand $x \in C$, elle peut être considérée comme une transformation linéaire et totalement continue de l'espace C en lui-même (3). En tenant compte du fait que la convergence dans C est plus forte que dans L, il en résulte que l'opération $U_\varepsilon(x)$ considérée comme une transformation linéaire de l'espace L, a le carré totalement continu.

La continuité totale de l'opération $U(x)$ en résulte en vertu de l'évaluation (3), d'après un théorème connu (4).

Les méthodes exposées ci-dessus peuvent servir à étudier la classe des transformations $U(x)$

$$U(x) = \int_0^s \frac{K(s, t)}{(s - t)^{\frac{m-1}{m}}} x(t)\, dx, \tag{II}$$

où m est un entier positif et

$$\underset{\substack{0 \leqslant s \leqslant 1 \\ 0 \leqslant t \leqslant s}}{\text{vrai max}} |K(s, t)| = K_0 < \infty.$$

Les deux propositions suivantes ont lieu:

1) parmis les transformations (II) il y a des transformations non totalement continues,

2) la puissance $(m + 1)$-ième de la transformation (II) est totalement continue.

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MATHÉMATIQUES APPLIQUÉES

SUR LES TABLES DES PRODUITS D'ÉTENDUE MINIMUM

Par L. NEUSCHULER

(Présenté par I. M. Vinogradow, de l'Académie, le 17. XII. 1937)

Nous introduisons dans cette note quelques caractéristiques des tables qui permettent de construire des tables d'étendue minimum.

Appelons somme conjointe des nombres $a_1, a_2, \dots a_k$ le nombre formé en inscrivant l'un après l'autre les nombres indiqués ci-dessus

$$\boxed{a_1 a_2 \dots a_k}.$$

Nous dirons que la table suivante

$$\left.\begin{array}{l} a_{11}\ a_{12} \dots\ a_{1m_1} \\ A_{11} A_{12} \dots A_{1m_1} \\ a_{21}\ a_{22} \dots\ a_{2m_2} \\ A_{21} A_{22} \dots A_{2m_2} \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ a_{k1}\ a_{k2} \dots\ a_{km_k} \\ A_{k1} A_{k2} \dots A_{km_k} \end{array}\right\} \qquad (I)$$

est une table d'ordre k de la fonction $f(x)$ dans (a, b), quand pour tout x compris en (a, b) nous pouvons extraire de notre table k tels éléments

$$a_{i_1 j_1}, a_{i_2 j_2}, \dots a_{i_k j_k} \text{ que } x = \boxed{a_{i_1 j_1} a_{i_2 j_2} \dots a_{i_k j_k}} \text{ et } f(x) = \boxed{A_{i_1 j_1} A_{i_2 j_2} \dots A_{i_k j_k}},$$

si on ne peut diminuer son ordre sans augmenter le nombre d'éléments.

Notre table a une dimension si pour tout $a \leqslant x \leqslant b$, $i_1 = i_2 = \dots = i_k$. Par la transposition des éléments compris dans une ligne on peut évidemment transformer la table de la façon suivante:

$$\left|\begin{array}{l} a_{i11}\ a_{i12} \dots\ a_{i1m_{i1}} \\ A_{i11} A_{i12} \dots A_{i1m_{i1}} \end{array}\right| \left|\begin{array}{l} a_{i21}\ a_{i22} \dots\ a_{i2m_{i2}} \\ A_{i21} A_{i22} \dots A_{i2m_{i2}} \end{array}\right| \dots \left|\begin{array}{l} a_{ik1}\ a_{ik2} \dots\ a_{ikm_{ik}} \\ A_{ik1} A_{ik2} \dots A_{ikm_{ik}} \end{array}\right| \quad i = 1, 2, \dots M.$$

Alors nous avons toujours: $x = \boxed{a_{i1p_1} a_{i2p_2} \dots a_{ikp_k}}$.

L'ensemble des a_{ipl} ($i=1, 2, \ldots M$; $l=1, 2, \ldots m_{ip}$) forme la p-me section de la table (I). La table d'ordre k contient ainsi k sections.

La ligne de la variable indépendante contenant «a» sera dite initiale, celle contenant «b» sera finale.

Nous dirons que les lignes d'une section sont égales, si les éléments à l'aide desquels nous formons les valeurs de l'argument y sont les mêmes. Laissons dans chaque section de la table (1) les lignes inégales et appelons cette nouvelle table base de la table (1).

La table à une dimension est homogène si toutes les lignes de sa base, sauf peut être la ligne finale, contiennent le même nombre d'éléments.

Nous nous occupons dans cette note des tables homogènes à une dimension.

Appelons étendue de la table d'ordre k l'expression $M\sum_{i=1}^{k} m_i n_i$,

où M est le nombre de lignes, m_i—le nombre d'éléments de la ligne initiale de la i-me section, $n_i = 1 + E\lg A_i$, et A_i est le plus grand élément de la i-me section.

En divisant l'étendue de la table du premier ordre de $f(x)$ dans (a, b) par l'étendue de notre table nous obtenons le degré de tabulation de cette dernière. Notre table est optimale dans (a, b) si son degré de tabulation n'est pas moins que celui de toute autre table du même ordre de $f(x)$ dans (a, b).

Nous dirons que les nombres $a_1 a_2 \ldots a_k$ s'additionnent conjointement avec le décalage «c», si «c» est le plus grand des c_i qu'il faut ajouter à a_i avant d'inscrire à son côté a_{i+1}. Dans une table avec le décalage «c» les éléments qui forment les valeurs de la fonction s'additionnent conjointement avec le décalage «c» dans $R \leqslant M-1$ lignes.

Convenons d'écrire les nombres dans l'intervalle $(0, 10^m-1)$ * en les complétant jusqu'à m chiffres à l'aide des zéros ajoutés à gauche.

Les lemmes suivants ont lieu pour la fonction ax définie dans l'intervalle $0, 10^m-1$ ($10^{n-1}+1 \leqslant a \leqslant 10^n-1$).

Nous nous y occupons de sa table optimale d'ordre k avec le décalage 1.

Lemme 1. L'égalité $E\ \lg A_{ipl} = E\lg A_{jpr} = E\lg a_{qps}$ a lieu pour $p \geqslant 2$ où $A_{ipl}, A_{jpr}, a_{qps}$ sont respectivement les plus grands des éléments $A_{ip1}, A_{ip2}, \ldots; A_{jp1}, A_{jp2}, \ldots; a_{qp1}, a_{qp2}, \ldots$ ($q = 1, 2, \ldots M$).

Lemme 2. Les éléments de la 1-ère section d'une ligne quelconque de la table sont tous les nombres de l'intervalle $(0, P)$ où P est un nombre $10^{n_1}-1$.

Lemme 3. Les éléments des lignes de la s-me section de la base (sauf peut être ceux de la ligne finale) ($s \geqslant 2$) sont tous nombres entiers de l'intervalle $x_{is}, x_{is} + E\frac{10^{n_s}}{a}$ ($i = 1, 2, \ldots M-1$), où $n_s = 1 + E\lg a_{isl}$ (a_{isl} est le plus grand des $a_{is1}, a_{is2}, \ldots$) et x_{is} sont les éléments de la progression arithmétique: zéro, $1 + E\frac{10^{n_s}}{a}, \ldots$

Quand $i = M$ (ligne finale) l'intervalle se transformera en

$$\left[(M-1)\left(1 + E\frac{10^{n_s}}{a}\right),\ 10^{n_s}-1\right].$$

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* Nous nous servons dans les lignes qui suivent du système décimal de la numération quoique les théorèmes 1 et 2 que nous formulons dans la suite restent valables pour tout système de numération sauf le système binaire.

Les lemmes 1—3 nous permettent d'obtenir l'expression suivante pour le degré de tabulation de la table du k-me ordre avec le décalage 1:

$$\frac{10^{n_1}\prod_{s=2}^{k}\left(1+E\frac{10^{n_s}}{a}\right)(m+n)}{10^{n_1}(n+n_1)+\sum_{s=2}^{k}\left(1+E\frac{10^{n_s}}{a}\right)n_s}. \qquad (A)$$

Pour l'existence de la table conjointe* du k-me ordre il faut et il suffit que $m \geqslant (k-1)n+1$.

Par conséquent, la détermination de la structure de la table optimale aboutit à une telle décomposition $m \geqslant (k-1)n+1$ en k composantes: $n_1, n_2, \ldots n_k$, où l'expression (A) est maximale.

Lorsque les inégalités $1 \leqslant q \leqslant p$ et $10^{n-1}+1 \leqslant a \leqslant 10^n-1$ sont vérifiées, on trouve facilement les relations suivantes:

$$\frac{1+E\frac{10^{n+p}}{a}}{10^q\left(1+E\frac{10^{n+p-q}}{a}\right)} > \frac{1}{2}, \qquad (1)$$

$$\frac{10^{p+1+q}(n+p+q+1)+\left(1+E\frac{10^{n+p-q}}{a}\right)(n+p-q)}{10^{p+1}(n+p+1)+\left(1+E\frac{10^{n+p}}{a}\right)(n+p)} > 5, \qquad (2)$$

$$\frac{10^q\left(1+E\frac{10^{n+p}}{a}\right)}{1+E\frac{10^{n+q+p}}{a}} \geqslant 1, \qquad (3)$$

$$10^{p+1-q}(n+p+1-q)+\left(1+E\frac{10^{n+p+q}}{a}\right)(n+p+q)-$$

$$-10^{p+1}(n+p+1)-\left(1+E\frac{10^{n+p}}{a}\right)(n+p) \geqslant 1, \qquad (4)$$

d'où on a:

$$\frac{\left(1+E\frac{10^{n+p}}{a}\right)\left[10^{p+1+q}(n+p+1+q)+\left(1+E\frac{10^{n+p-q}}{a}\right)(n+p-q)\right]}{10^q\left(1+E\frac{10^{n+p-q}}{a}\right)\left[10^{p+1}(n+p+1)+\left(1+E\frac{10^{n+p}}{a}\right)(n+p)\right]} > 1, \qquad (5)$$

$$\frac{10^q\left(1+E\frac{10^{n+p}}{a}\right)\left[10^{p+1-q}(n+p+1-q)+\left(1+E\frac{10^{n+p+q}}{a}\right)(n+p+q)\right]}{\left(1+E\frac{10^{n+p+q}}{a}\right)\left[10^{p+1}(n+p+1)+\left(1+E\frac{10^{n+p}}{a}\right)(n+p)\right]} > 1. \qquad (6)$$

Il en résulte à l'aide des lemmes 1—2:

Théorème 1. Soit T une table optimale du deuxième ordre avec le décalage 1 construite pour la fonction ax. Quand $m-n=2p+1$ la ligne initiale * de l'argument se présente sous la forme:

I section	II section
$0 \leqslant x_1 \leqslant 10^{p+1}-1$	$0 \leqslant x_2 \leqslant E\frac{10^{n+p}}{a}$

(où a est un nombre à n chiffres; $1 \leqslant x \leqslant 10^m-1$).

En ajoutant les formules (1), (3) nous aurons pour $1 \leqslant q \leqslant p-1$:

$$10^{p-q}(n+p-q)+\left(1+E\frac{10^{n+p+q}}{a}\right)(n+p+q)-$$

$$-\left(1+10^{p}+E\frac{10^{n+p}}{a}\right)(n+p)>0, \qquad (7)$$

et pour $2\leqslant q\leqslant p$:

$$\frac{10^{p+q}(n+p+q)+\left(1+E\frac{10^{n+p-q}}{a}\right)(n+p-q)}{\left(1+10^{p}+E\frac{10^{n+p}}{a}\right)(n+p)}>5, \qquad (8)$$

d'où pour $1\leqslant q\leqslant p-1$:

$$\frac{10^{q}\left(1+E\frac{10^{n+p}}{a}\right)\left[10^{p-q}(n+p-q)+\left(1+E\frac{10^{n+p+q}}{a}\right)(n+p+q)\right]}{\left(1+E\frac{10^{n+p+q}}{a}\right)\left(1+10^{p}+E\frac{10^{n+p}}{a}\right)(n+p)}>1, \qquad (9)$$

et pour $2\leqslant q\leqslant p$:

$$\frac{\left(1+E\frac{10^{n+p}}{a}\right)\left[10^{p+q}(n+p+q)+\left(1+E\frac{10^{n+p-q}}{a}\right)(n+p-q)\right]}{10^{q}\left(1+E\frac{10^{n+p-q}}{a}\right)\left(1+10^{p}+E\frac{10^{n+p}}{a}\right)(n+p)}>1. \qquad (10)$$

Les lemmes 1—3 nous donnent le théorème suivant:

Théorème 2. Quand $m-n=2p$ la ligne initiale d'argument d'une table optimale de deuxième ordre avec le décalage 1 pour ax se présente sous l'une des deux formes suivantes:

I section	II section
$0\leqslant x_1\leqslant 10^{p}-1$	$0\leqslant x_2\leqslant E\frac{10^{n+p}}{a}$
$0\leqslant x_1\leqslant 10^{p+1}-1$	$0\leqslant x_2\leqslant E\frac{10^{n+p-1}}{a}$

Il est évident que l'ensemble des tables pour $y=ax$ $(1\leqslant x\leqslant 10^{m}-1)$ pour tout «a» compris dans l'intervalle $(10^{n-1}+1,\ 10^{n}-1)$ nous donne la table de $z=xy$ $(1\leqslant x\leqslant 10^{m}-1;\ 1\leqslant y\leqslant 10^{n}-1)$.

Si nous exigeons l'homogénité de cette dernière, sa structure sera définie par la ligne initiale de $(10^{n}-1)x$.

Puisque dans le cas où $a=10^{n}-1$ l'égalité (10) a lieu pour $q=1$, nous pouvons en conservant l'optimum pour $m-n=2p+1$ et pour $m-n=2p$, transposer les lignes de la table $z=xy$ de façon que la page initiale de la table (c'est-à-dire, l'ensemble des lignes initiales des tables pour les divers y) se présente sous la forme suivante:

y \ x	$10^{m-n-p-1}\leqslant x_1\leqslant 10^{m-n-p}-1$	$0\leqslant x_2\leqslant E\frac{10^{n+p}}{a}$
$10^{n-1}+1$		
$10^{n-1}+2$**		
. . . . .		
$10^{n}-1$		

qui est la mieux appropriée pour les tables des produits.

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* La structure de la table homogène est définie par la structure de sa ligne initiale.

** Comme la ligne de l'argument dans la 1-ère et dans la 2-ème section est la même pour tout y, elle n' est donnée q'une fois au commencement.

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PHYSIQUE

SUR LA TRANSITION DE L'OPTIQUE ONDULATOIRE À L'OPTIQUE GÉOMÉTRIQUE

Par S. M. RYTOV

(Présenté par L. I. Mandelstam, de l'Académie, le 20. XI. 1937)

On considère généralement la transition à l'approximation de l'optique géométrique à l'aide de l'équation d'onde scalaire. On en déduit en approximation d'ordre nul par rapport à la longueur d'onde λ l'équation de l'eiconal et en première approximation—la loi de conservation du flux lumineux. Cependant, une théorie électromagnétique suivie devrait opérer avec les équations vectorielles de Maxwell *.

Il faut s'attendre alors à ce que l'étude de la première approximation donne non seulement une loi pour l'intensité, mais aussi une loi pour la polarisation, autrement dit qu'elle permettra de conclure comment les vecteurs $\mathbf{E}$ et $\mathbf{H}$ tournent autour du rayon lorsque l'on glisse le long de ce dernier.

Il est démontré dans ce travail que les conditions d'existence de la première approximation permettent en effet de déduire non seulement la loi de conservation du flux lumineux, mais aussi une loi très simple pour la rotation des vecteurs du champ.

Les équations de Maxwell dans un milieu diélectrique (nous posons la perméabilité magnétique égale à l'unité) dans le cas d'une onde monochromatique, c'est-à-dire pour $\mathbf{E}$ et $\mathbf{H}$ proportionnels à $e^{i\omega t}$, ont la forme:

$$\begin{aligned} \operatorname{rot}\mathbf{E} &= -ik\mathbf{H}, \\ \operatorname{rot}\mathbf{H} &= ik\varepsilon(x, y, z)\,\mathbf{E}, \end{aligned} \tag{1}$$

ε étant la constante diélectrique du milieu et

$$k = \frac{\omega}{c} = \frac{2\pi}{\lambda}.$$

Nous cherchons une solution de la forme:

$$\mathbf{E} = \frac{\mathbf{A}(x, y, z)}{\sqrt{\varepsilon}} \cdot e^{-ik\phi(x, y, z)}$$

$$\mathbf{H} = \mathbf{B}(x, y, z) \cdot e^{-ik\phi(x, y, z)}$$

En substituant cette solution dans (1), nous obtenons:

$$\begin{aligned} [\nabla\phi, \mathbf{B}] + \sqrt{\varepsilon}\,\mathbf{A} &= -\frac{i}{k}\operatorname{rot}\mathbf{B}, \\ -\sqrt{\varepsilon}\,\mathbf{B} + [\nabla\phi, \mathbf{A}] &= -\frac{i}{k}\left(\operatorname{rot}\mathbf{A} - \frac{1}{2\varepsilon}[\nabla\varepsilon, \mathbf{A}]\right). \end{aligned} \tag{2}$$

* Cela a été effectué pour le cas d'un milieu homogène par V. S. Ignatovsky. Travaux de l'Institut d'Optique, I, 3 (1920); v. aussi V. A. Fock, ibid, III, 27 (1924).

Nous développons maintenant $\mathbf{A}$ et $\mathbf{B}$ en séries d'après les puissances de $\frac{i}{k}$, en supposant que ces séries convergent:

$$\mathbf{A} = \mathbf{A}_0 + \frac{i}{k}\mathbf{A}_1 + \dots,$$
$$\mathbf{B} = \mathbf{B}_0 + \frac{i}{k}\mathbf{B}_1 + \dots$$

En substituant ces séries dans (2) et en comparant les termes qui contiennent les mêmes puissances de $\frac{i}{k}$, nous obtenons les équations d'approximation d'ordre nul:

$$\begin{aligned} [\nabla\psi, \mathbf{B}_0] + \sqrt{\varepsilon}\,\mathbf{A}_0 &= 0, \\ -\sqrt{\varepsilon}\,\mathbf{B}_0 + [\nabla\psi, \mathbf{A}_0] &= 0, \end{aligned} \tag{3}$$

et les équations de première approximation:

$$\begin{aligned} [\nabla\psi, \mathbf{B}_1] + \sqrt{\varepsilon}\,\mathbf{A}_1 &= -\operatorname{rot}\mathbf{B}_0, \\ -\sqrt{\varepsilon}\,\mathbf{B}_1 + [\nabla\psi, \mathbf{A}_1] &= -\left(\operatorname{rot}\mathbf{A}_0 - \frac{1}{2\varepsilon}[\nabla\varepsilon, \mathbf{A}_0]\right). \end{aligned} \tag{4}$$

Les équations homogènes (3) ont une solution non nulle sous la condition que le déterminant du système est nul. Il est aisé de voir que cette condition est l'équation de l'eiconal:

$$(\nabla\psi)^2 = \varepsilon.$$

Posant:

$$\nabla\psi = \sqrt{\varepsilon}\,\mathbf{t},$$

$\mathbf{t}$ étant le vecteur unitaire de la normale à l'onde, nous amenons les équations (3) et (4) à la forme:

$$\left.\begin{aligned} [\mathbf{t}, \mathbf{B}_0] + \mathbf{A}_0 &= 0, \\ -\mathbf{B}_0 + [\mathbf{t}, \mathbf{A}_0] &= 0; \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} [\mathbf{t}, \mathbf{B}_1] + \mathbf{A}_1 &= -\frac{1}{\sqrt{\varepsilon}}\operatorname{rot}\mathbf{B}_0 = \mathbf{X}, \\ -\mathbf{B}_1 + [\mathbf{t}, \mathbf{A}_1] &= -\frac{1}{\sqrt{\varepsilon}}\left(\operatorname{rot}\mathbf{A}_0 - \frac{1}{2\varepsilon}[\nabla\varepsilon, \mathbf{A}_0]\right) = \mathbf{Y}. \end{aligned}\right\} \tag{6}$$

Les équations (5) ont deux et seulement deux solutions particulières linéairement indépendantes, correspondant à deux directions de polarisation:

$$\left.\begin{aligned} \mathbf{A}_0 &= \mathbf{n}, \\ \mathbf{B}_0 &= \mathbf{b}, \end{aligned}\right\} \quad \text{et} \quad \left.\begin{aligned} \mathbf{A}_0 &= \mathbf{b}, \\ \mathbf{B}_0 &= -\mathbf{n}, \end{aligned}\right\} \tag{7}$$

où les vecteurs $\mathbf{n}$ et $\mathbf{b}$, que nous pouvons considérer comme des vecteurs unitaires, ne satisfont qu'à la seule condition qu'ils forment avec $\mathbf{t}$ un trièdre orthogonal. Pour fixer les idées, posons que $\mathbf{n}$ est le vecteur unitaire de la normale principale du rayon; alors $\mathbf{b}$ sera le vecteur unitaire de la binormale. La solution générale de l'équation (5) aura la forme:

$$\begin{aligned} \mathbf{A}_0 &= \Phi_1\mathbf{n} + \Phi_2\mathbf{b}, \\ \mathbf{B}_0 &= \Phi_1\mathbf{b} - \Phi_2\mathbf{n}, \end{aligned}$$

Φ_1 et Φ_2 étant des fonctions arbitraires.

Nous avons donc en approximation d'ordre nul un trièdre orthogonal $\mathbf{A}_0$, $\mathbf{B}_0$, $\mathbf{t}$ dans chaque point du rayon (de la ligne t); mais les fonctions Φ_1 et Φ_2 étant encore indéterminées, ce trièdre n'est en aucune façon lié aux trièdres pareils situés dans d'autres points du rayon. Autrement dit, la variation de l'intensité et la rotation des vecteurs $\mathbf{A}_0$ et $\mathbf{B}_0$ sont encore inconnues. L'une et l'autre doivent se révéler des conditions d'existence d'une solution du système d'équations inhomogènes de première approximition (6), dont le déterminant est nul. Ces conditions consistent, comme l'on sait, dans l'orthogonalité des parties droites de (6) à chacune des solutions linéairement indépendantes du système homogène transposé. Mais il est facile de voir que le tableau des équations homogènes (5) est anti-symétrique et que par conséquent les équations du système transposé ne diffèrent des équations du système initial que par le signe commun. Il en suit que les parties droites des équations (6) doivent être orthogonales simplement à chacune des solutions (7) du système initial. Cela donne deux conditions:

$$(\mathbf{X}, \mathbf{b}) + (\mathbf{Y}, \mathbf{n}) = 0,$$
$$-(\mathbf{X}, \mathbf{n}) + (\mathbf{Y}, \mathbf{b}) = 0,$$

où l'on doit substituer dans $\mathbf{X}$ et $\mathbf{Y}$ la solution générale du système homogène.

En effectuant cette substitution, l'on obtient après quelques transformations évidentes:

$$\left.\begin{aligned} &\Phi_1 (\mathbf{n}\,\mathrm{rot}\,\mathbf{n} + \mathbf{b}\,\mathrm{rot}\,\mathbf{b}) - \Phi_2 \,\mathrm{div}\,\mathbf{t} - 2(\mathbf{t}, \nabla\Phi_2) + \frac{\Phi_2}{2\varepsilon}(\mathbf{t}, \nabla\varepsilon) = 0, \\ &\Phi_2 (\mathbf{n}\,\mathrm{rot}\,\mathbf{n} + \mathbf{b}\,\mathrm{rot}\,\mathbf{b}) + \Phi_1 \,\mathrm{div}\,\mathbf{t} + 2(\mathbf{t}, \nabla\Phi_1) - \frac{\Phi_1}{2\varepsilon}(\mathbf{t}, \nabla\varepsilon) = 0. \end{aligned}\right\} \quad (8)$$

En multipliant la première équation par $-\frac{\Phi_2}{\sqrt{\varepsilon}}$, la seconde par $\frac{\Phi_1}{\sqrt{\varepsilon}}$ et en faisant l'addition, nous trouvons la loi de variation de l'amplitude:

$$\mathrm{div}\left(\frac{\Phi_1^2 + \Phi_2^2}{\sqrt{\varepsilon}} \cdot \mathbf{t}\right) = 0.$$

En multipliant la première équation par Φ_1, la seconde par Φ_2 et en additionnant de nouveau, nous obtenons:

$$\frac{1}{2}(\mathbf{n}\,\mathrm{rot}\,\mathbf{n} + \mathbf{b}\,\mathrm{rot}\,\mathbf{b}) = \frac{(\mathbf{t}, \Phi_1\nabla\Phi_2 - \Phi_2\nabla\Phi_1)}{\Phi_1^2 + \Phi_2^2} = \left(\mathbf{t}, \nabla \,\mathrm{arctg}\,\frac{\Phi_2}{\Phi_1}\right) = (\mathbf{t}, \nabla)\varphi, \quad (9)$$

$\varphi = \mathrm{arctg}\frac{\Phi_2}{\Phi_1}$ étant l'angle compris entre $\mathbf{A}_0$ et la normale principale.

Profitons maintenant de l'expression:

$$-\mathbf{t}\,\mathrm{rot}\,\mathbf{t} + \mathbf{n}\,\mathrm{rot}\,\mathbf{n} + \mathbf{b}\,\mathrm{rot}\,\mathbf{b} = \frac{2}{T}\,*,$$

T étant le rayon de torsion.

Dans le cas considéré, $\mathbf{t}$ indique la direction du vecteur potentiel $\left(\mathbf{t} = \frac{1}{\sqrt{\varepsilon}}\nabla\psi\right)$, donc $\mathbf{t}\,\mathrm{rot}\,\mathbf{t} = 0$ et par conséquent

$$\mathbf{n}\,\mathrm{rot}\,\mathbf{n} + \mathbf{b}\,\mathrm{rot}\,\mathbf{b} = \frac{2}{T}. \quad (10)$$

Substituant cela dans (9) et remarquant que

$$(\mathbf{t}, \nabla) = \frac{dx}{ds}\frac{\partial}{\partial x} + \frac{dy}{ds}\frac{\partial}{\partial y} + \frac{dz}{ds}\frac{\partial}{\partial z} = \frac{d}{ds}, \quad (11)$$

* Voir par exemple R. Rothe. Anwendungen der Vektoranalysis auf Geometrie. Jahresber. d. deutsch. Math. Ver. XXI, 257, 1912.

s étant l'arc pris le long de la courbe (dans le cas considéré—le long du rayon), nous obtenons le résultat final sous la forme:

$$\frac{d\varphi}{ds} = \frac{1}{T}. \tag{12}$$

Par conséquent, si le rayon possède une torsion, **E** et **H** tournent par rapport au trièdre intrinsèque et cette rotation est d'autant plus grande que la torsion est plus grande. **E** et **H** ne sont immobiles par rapport au trièdre intrinsèque que dans le seul cas d'absence de torsion, autrement dit dans le cas où le rayon est une courbe plane.

Il est facile de calculer encore les vitesses absolues de la rotation des vecteurs unitaires **e** et **h**, indiquant les directions de **E** et **H**:

$$\begin{aligned} \mathbf{e} &= \mathbf{n}\cos\varphi + \mathbf{b}\sin\varphi, \\ \mathbf{h} &= \mathbf{b}\cos\varphi - \mathbf{n}\sin\varphi. \end{aligned} \tag{13}$$

Profitant de (12) et des formules de Frenet:

$$\frac{d\mathbf{t}}{ds} = \frac{\mathbf{n}}{R}, \quad \frac{d\mathbf{n}}{ds} = -\frac{\mathbf{t}}{R} - \frac{\mathbf{b}}{T}, \quad \frac{d\mathbf{b}}{ds} = \frac{\mathbf{n}}{T},$$

R étant le rayon de courbure, nous trouvons:

$$\frac{d\mathbf{e}}{ds} = -\frac{\cos\varphi}{R}\mathbf{t}, \quad \frac{d\mathbf{h}}{ds} = \frac{\sin\varphi}{R}\mathbf{t}.$$

Par conséquent, la rotation de chacun des deux vecteurs **E** et **H** relativement à un système de coordonnées indépendant, lors de la croissance de s, cesse auprès de ceux des points du rayon où ce vecteur est colinéaire à la binormale.

Il est facile enfin de déduire à l'aide des formules (10), (12) et (13) que la somme

$$\mathbf{e}\,\mathrm{rot}\,\mathbf{e} + \mathbf{h}\,\mathrm{rot}\,\mathbf{h} = 0 \tag{14}$$

à courbure et torsion arbitraires, mais que $\mathbf{e}\,\mathrm{rot}\,\mathbf{e}$ et $\mathbf{h}\,\mathrm{rot}\,\mathbf{h}$ ne sont pas en général nuls *.

Il n'existe donc pas en général de système triorthogonal de surfaces, dont les trajectoires orthogonales seraient les lignes de **E**, **H** et **t**.

La condition nécessaire et, selon (14) suffisante, d'existence d'un tel système serait $\mathbf{e}\,\mathrm{rot}\,\mathbf{e} = 0$.

On peut considérer comme d'ordinaire la rotation du plan de polarisation, exprimée par la formule (12), comme le résultat d'une différence de vitesses des ondes composantes polarisées circulairement. Il est facile de voir que les vitesses de propagation des ondes polarisées circulairement de rotation droite et gauche seront:

$$v_{dr.,g} = \frac{c}{\sqrt{\varepsilon} + \frac{\lambda}{2\pi T}}$$

Je profite de l'occasion pour exprimer ma reconnaissance au prof. M. Léontovitch pour ses conseils précieux.

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* $\mathbf{e}\,\mathrm{rot}\,\mathbf{e} = -\mathbf{h}\,\mathrm{rot}\,\mathbf{h} = \gamma$, γ étant ce qu'on nomme la torsion géodésique des lignes **e**, placées sur les surfaces $\psi = \mathrm{const}$.

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CHEMICAL PHYSICS

THE PRODUCTION OF DETONATION IN EXPLOSIVES UNDER THE ACTION OF A THERMAL IMPULSE

By. A. F. BELAJEV

(Communicated by N. N. Semenov, Member of the Academy, 15. XII. 1937)

It is often necessary in practice to deal with the action of some form of thermal impulse on an explosive. In particular, as has been shown by J. B. Chariton, local heating in a minute volume (a few cubic microns) should occur as the result of a blow in both liquid and solid explosives, and the temperature attained will considerably exceed the explosion temperature, so that it is natural to connect the production of detonation by shock with the appearance of these places of local high temperature. The present work was carried out with the object of establishing at least the general laws of thermal initiation in the case when the thermal impulse is concentrated in a volume of the order of a few cubic microns in a very short interval of time (10^{-3}—10^{-4} sec.). Platinum wires with 2 and 5 μ diameters and 3 mm long situated in the explosive, were used as the source of the thermal impulse. The wire was heated by an electric current obtained by closing an electric circuit containing a charged condenser, and the presence or absence of detonation was recorded. The total duration of the thermal impulse did not exceed 10^{-4} sec. The maximum temperature of the wire was calculated on the assumption that the heating takes place adiabatically and that all the energy of the condenser is liberated in the wire at the moment of discharge. The liquid explosives—nitrogen chloride and trinitroglycerine, were used.

Experimental Results

Nitrogen Chloride

1) $d = 5\,\mu$; $C = 10^{-5}$ F; $E = 120$ V; $W = 1.7 \cdot 10^{-2}$ cal.
2) $d = 5\,\mu$; $C = 5 \cdot 10^{-7}$F; $E = 200$ V; $W = 2.4 \cdot 10^{-3}$ cal.
3) $d = 2\,\mu$; $C = 5 \cdot 10^{-7}$F; $E = 50$ V; $W = 1.5 \cdot 10^{-4}$ cal.
4) $d = 2\,\mu$; $C = 10^{-9}$ F; $E = 500$ V; $W = 3 \cdot 10^{-5}$ cal.

Here d is the diameter of the wire, C—the capacity of the condenser, E—the minimum potential across the condenser at which a detonation is produced, W—the minimum energy necessary for the production of a detonation ($W = 0.24 \frac{CE^2}{2}$ cal.).

It is clear from these data that the minimum energy required for the production of a detonation decreases with the diameter of wire, as well as with the time of discharge (decreasing capacity). The lowest energy obserw

ed, $3 \cdot 10^{-5}$ cal., corresponds to a «theoretical» heating of the wire to 5 000° C, i. e. the platinum wire is melted and probably evaporated.

In the case of trinitroglycerine a potential of 40 000 V at a capacity of $5 \cdot 10^{-7}$ F corresponding to a total energy of the order of 100 cal., which is sufficient to convert a wire of 0·1 mm diameter in 10^{-5}—10^{-6} sec. into vapour at a temperature of about 20 000° C, was nevertheless incapable of producing detonation or explosion. This result indicates that either the hypothesis of local heating as the cause of shock detonation is incorrect or that the conditions of thermal initiation in our method were not identical with those occurring when an explosive is detonated by a blow.

An analysis of the phenomena occurring during thermal initiation showed that it was possible, under the conditions of our experiments, that some of the energy liberated was absorbed as a result of evaporation. In the case of most secondary explosives the activation energy is greater than the energy of evaporation, as a result of which the number of evaporating molecules may greatly exceed the number of those reacting. It is true that evaporation can only occur from the surface, whilst reaction can take place throughout the volume. However, when an explosion is set off by means of a flame or a heated body etc., the energy is given to the surface layer and this facilitates the dissipation of this energy by evaporation. It can be shown by calculation that for molecular surface layer of trinitroglycerine at 160° C, the number of evaporated molecules is roughly 10^6 times greater than the number of reacting in the same layer. In this way, under favourable circumstances the evaporation may absorb both the energy supplied from without as well that liberated as a result of the reaction. It is, moreover, clear that as soon as the temperature rises above the boiling point of the given explosive in the element in which the local heating due to the wire occurs, evaporation will take place even in the volume and will prevent any further rise of temperature.

An increase of pressure under which the experiments are conducted should clearly reduce the tendency to evaporate (or more correctly reduce the diffusion—the escape of the evaporated molecules) and should also rase the boiling point. Experiments were therefore conducted at a static pressure of 100 atm.

Experimental Results at 100 atm

Nitrogen Chloride

1) $d = 2\,\mu$; $C = 5 \cdot 10^{-7}$ F; $E = 12$ V; $W = 9 \cdot 10^{-6}$ cal.
2) $d = 2\,\mu$; $C = 10^{-9}$ F; $E = 160$ V; $W = 3 \cdot 10^{-6}$ cal.
3) $d = 2\,\mu$; $C = 5 \cdot 10^{-11}$ F; $E = 400$ V; $W = 9.6 \cdot 10^{-7}$ cal.

Thus we see from these data that for a given capacity the minimum energy required at the high pressure is roughly 10 times less than that at atmospheric pressure, so that a thermal impulse containing $9.6 \cdot 10^{-7}$ cal. liberated by a wire 2 μ thick at a pressure of 100 atm is sufficient to cause detonation in nitrogen chloride. In this case the maximum temperature of the wire is 170° C and the total duration of the thermal action does not exceed 10^{-4} sec.

In the case of trinitroglycerine, with $C = 5 \cdot 10^{-7}$ F and E = 4 000 V, one experiment out of seven gave rise to an undoubted detonation.

Clearly, when an explosive substance is struck by a blow, a considerable pressure is produced. This pressure in the case of the minimum shock required for producing detonation in nitrogen chloride is of the order of 200—300 atm, whilst in trinitroglycerine it is of the order of 10 000 atm. It follows that the conditions used for the experiment with thermal initia-

tion of nitrogen chloride at a pressure of 100 atm are identical with those of shock detonation in the same explosive. In the case of nitroglycerine this pressure is not sufficient. Thus, even when the local heating takes place in minute volume elements, and over a very short time interval detonation can develop if the pressure is sufficiently high. It is this condition which is satisfied in the case of shock detonation.

Concerning the Influence of Evaporation on the Development of Detonation in Explosives. The results recorded above lead to the conclusion that evaporation hinders the development of detonation in volatile explosives by absorbing a considerable amount of energy. Most primary explosives, even in the vicinity of the explosion temperature, are solids possessing very low volatility. The ease with which these substances detonate is attributable, of course, in the first place, to their peculiar chemical qualities, but detonation in these substances is also facilitated by the absence of loss of energy in evaporation. On the other hand, the majority of secondary explosives are liquids in the neighbourhood of the explosion temperature and possess considerable volatility.

It has already been shown that all the heat applied to a volatile explosive should be used up in evaporation. The temperature of the vapours of the explosive which are in contact with the source of heat increases until combustion starts, whereupon the heat of reaction gives rise to the evaporation of new portions of the substance and so on. Thus, the combustion cannot enter the condensed phase and we are led to the hypothesis that slow combustion of explosives should proceed in the gas phase. The evaporation process precedes the combustion process and the combustion zone is separated from the condensed phase by a particular type of barrier which absorbs the energy. Most explosives which possess an appreciable volatility are also capable of slow combustion. It is natural to suppose that in the case of explosives of the lead azide type or silver acetylide, the volatility of which is very low, combustion commences in the condensed phase and, being an unstable process, instantly goes over into detonation. It can be shown that if combustion really takes place in the gas phase, then when the combustion regime has been established a direct and comparatively cold zone should exist between the combustion region and the remainder of the explosive in the condensed phase. In this zone the vapours undergo a process preparatory to combustion and the zone itself should increase with decreasing pressure. A zone of this type has been observed during the combustion of nitroglycol at a pressure of 20—30 mm Hg. At this pressure the zone was 1—2 mm.

The transition from slow combustion to detonation may be regarded according to the above view as the transition of combustion from the gas phase to the condensed phase, which rapidly leads to detonation. It is obvious that this transition should be facilitated by an increase of pressure since this reduces the evaporation.

Summarizing briefly the results, it may be noted that while in the case of volatile explosives the chemical factors are of primary importance in the process of the development of detonation, a considerable part is also played by evaporation which, under favourable conditions, may change the whole character of the chemical process and in particular retard the development of detonation and cause the slow combustion to take place from beginning to end in the gas phase.

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CHEMIE

ÜBER DIE 1-PROPYL-2-METHYLVALERIANSÄURE UND DIE 1, 2-DIMETHYLKAPRONSÄURE

Von M. S. KONDAKOWA und M. M. KATZNELSON

(Vorgelegt von A. E. Faworskij, Mitglied der Akademie, d. 11. XII. 1937)

Zur Erforschung der Struktur der aus dem Erdöl erhaltenen Säuren synthetisierten wir, zum Vergleich mit synthetisch erhaltenen Säuren, 1-Propyl-2-Methylvaleriansäure und 1.2-Dimethylkapronsäure.

Über 1-Propyl-2-Methylvaleriansäure

$$\begin{array}{l} CH_3 \cdot CH_2 \cdot CH - CH \cdot COOH \\ \qquad\qquad\quad | \qquad\quad | \\ \qquad\qquad CH_3 \quad C_3H_7 \end{array}$$

Diese Säure wurde durch Einwirkung des sekundären Butylbromids $CH_3 \cdot CH_2 \cdot CHBr \cdot CH_3$ auf Natriumpropylmalonester und Abspaltung von CO_2 von der Propyl-sek. Butylmalonsäure, die sich bei der Verseifung des Diäthyläthers dieser Säure bildet, erhalten:

$$\begin{array}{l} CH_3 \cdot CH_2 \cdot CH \cdot C \begin{array}{l} \diagup COOC_2H_5 \\ \diagdown COOC_2H_5 \end{array} \\ \qquad\qquad\quad | \qquad | \\ \qquad\qquad CH_3 \; C_3H_7 \end{array}$$

Gewinnung von Propyl-*sek.*-Butylmalonester. Aus 208 g Propylmalonester, in 250 g absolutem Alkohol gelöst, und 23.6 g Natrium wurde das Natriumderivat des Propylmalonesters hergestellt und zu demselben 160 g sekundären Butylbromids hinzugefügt. Das Gemisch wurde 8 Stunden lang im kochenden Wasserbad mit Rückflusskühler erwärmt. Hierauf wurde der Alkohol abdestilliert und Wasser zum Rückstand hinzugegeben. Die obere Schicht wurde von der Wasserschicht abgetrennt und letztere mit Äther extrahiert. Nach Trocknen mit geschmolzenem Kalziumchlorid und Abdestillation des Äthers wurde der Rückstand mit einem Dephlegmator fraktioniert.

Erhalten wurden folgende Fraktionen:

220—235°	65.0 g
235—245°	27.5 »
245—255°	98.0 »

Propyl-*sek.*-Butylmalonsäure. Die höchste Fraktion, die zwischen 245—255° (98 g) siedete, wurde mit einer Ätzkalilösung in

absolutem Alkohol beim Sieden einer Verseifung unterzogen. Hierbei fällt Kaliumsalz aus, das sich in wenig heissem Wasser löst; nach Hinzugabe von Säure wird ein Produkt ausgeschieden, wovon ein Teil nach einigen Tagen Stehen im Exsikkator fest wird. Nach Kristallisation aus Benzol scheiden sich Kristalle mit dem Schmelzpunkt 134—136° aus.

Ein Teil der Säure wurde durch Auflösung in schwachem Ammoniak und durch Ausfällung mit einer Silbernitratsalzlösung in ein Silbersalz verwandelt. Das erhaltene Silbersalz wurde bis zur Erreichung konstanten Gewichts im Vakuum-Exsikkator getroknet:

0.1018 g Sbst.: 0.0536 g Ag;
0.059 g Sbst.: 0.0310 g Ag.
$C_{10}H_{16}O_4Ag_2$. Ber.: Ag 51.9.
Gef.: Ag 52.64;
» : Ag 52.56.

1-Propyl-2-Methylvaleriansäure. Die Säure wurde aus der zweibasichen Säure durch Erwärmung bis zum Aufhören der Kohlensäureausscheidung erhalten. Die Säure wurde bei 233—234° destilliert:

0.0884 g Sbst.: 0.2217 g CO_2; 0.0916 g H_2O.
$C_9H_{18}O_2$. Ber.: C 68.35; H 11.39.
Gef.: C 68.40; H 11.60.
Gewicht der Säure bei 18° 1.0079 g
» des Wassers bei 18° 1.1059 g
» der Säure bei 0° 1.0239 g
» des Wassers bei 0° 1.1064 g
$d^{18}/_{18} = 0.9113$, $d\% = 0.9254$.

Methylester der 1-Propyl-2-Methyl-Valeriansäure wurde aus dem Silbersalz durch Kochen mit Methyljodid in absolutem Äther erhalten. Siedepunkt des Methylesters 183—184°.

0.1037 g Sbst; 0.2640 g CO_2; 0.1092 g H_2O.
$C_{10}H_{20}O_2$. Ber.: C 69.76; H 11.63.
Gef.: C 69.43; H 11.77.
Gewicht des Esters bei 18° 0.8839 g
» » Wassers » 18° 1.0061 »
» » Esters » 0° 0.8991 »
» » Wassers » 0° 1.0069 »
$d^{18}/_{18} = 0.8785$; $d\% = 0.8929$.

Das Säurechlorid wurde durch Erwärmen der Säure mit Phosphortrichlorid auf dem Wasserbad erhalten. Das Chlorid wurde im Vakuum bei 97—100° und 37 mm destilliert.

Das Säureamid wurde durch Einwirkung von Ammoniak auf eine Benzollösung des Chlorids erhalten. Das Amid stellt dünne schneeweisse Nadeln mit dem Schmelzpunkt 125° dar.

0.0946 g Sbst.: 7.58 cm³ N (19°.5, 753.6 mm);
0.1066 g Sbst.: 8.34 cm³ N (17°, 756.4 mm).
$C_9H_{19}ON$. Ber.: N 8.92. Gef.: N 9.07; N 8.99.

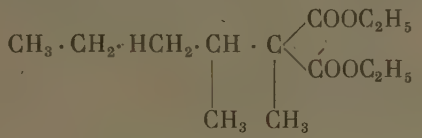
Das Säureanilid wurde durch Kochen im Wasserbad einer Benzollösung von Chloranhydrid mit einer Benzollösung von Anilin erhalten. Das Anilid stellt Nadeln mit dem Schmelzpunkt 110—111° dar.

0.1938 Sbst.: 10.46 cm³ N (20°, 754.3 mm).
$C_{15}H_{23}ON$. Ber.: N 6.01. Gef.: N 6.10.

Über die 1,2-Dimethylkapronsäure

$$CH_3 \cdot CH_2 \cdot CH_2 \cdot \underset{\displaystyle CH_3}{\underset{|}{CH}} \cdot \underset{\displaystyle CH_3}{\underset{|}{CH}} \cdot COOH$$

Diese Säure wurde durch Einwirkung von Methyljodid auf das Natriumderivat des Diäthylesters von 2-Methylpentan-1,1-Dikarbonsäure (1) und durch Abspaltung von CO_2 von 2-Methylpentan-1-Methyl-1,1-Dikarbonsäure, die sich bei der Verseifung des Diäthylesters dieser Säure bildet:



Der Diäthylester der 2-Methylpentan-1-Methyl-1,1-Dikarbonsäure. Aus 125 g Diäthylesters der 2-Methylpentan 1,1-Dikarbonsäure in 138 g abs. Alkohol und 12.5 g Natrium wurde das Natriumderivat hergestellt, zu demselben wurden 85 g Methyljodid hinzugegeben. Nach Sieden und entsprechender Behandlung wurden bei der Destillation bei 29 mm folgende Fraktionen erhalten:

120—140°	14 g
140—148°	100 g

Wiederholte Destillation der zweiten Fraktion bei 14 mm

120—128°	11 g
128—132°	85 »

2-Methylpentan-1-Methyl-1-1-Dikarbonsäure. Die höchste Fraktion mit dem Siedepunkt 128—132° bei 14 mm (85 g) wurde einer Verseifung unterzogen. Das Verseifungsprodukt stellt nach der Kristallisation aus Benzol Kristalle mit dem Schmelzpunkt 139—140°.

Ein Teil der Säure wurde in das Silbersalz verwandelt:

0.1977 g Sbst.: 0.1051 g Ag.
$C_{19}H_{14}O_4Ag_2$. Ber.: Ag 53.69. Gef.: Ag 53.18.

1, 2-Dimethylkapronsäure, Siedepunkt 223—224°.

0.1467 g Sbst.: 0.3581 g CO_2; 0.1482 g H_2O.
$C_8H_{16}O_2$. Ber.: C 66.66; H 11.11. Gef.: C 66.58; H 11.30

Gewicht	der Säure bei 18°	1.0215 g
»	des Wassers bei 18°	1.1059 »
»	der Säure » 0°	1.0363 »
»	des Wassers » 0°	1.1064 »

$d^{18}/_{18} = 0.9236$, $d\ \% = 0.9366$.

Aus der Säure wurde das Silbersalz erhalten

0.1888 g Sbst.: 0.0814 g Ag;
0.1199 g Sbst.: 0.0620 g Ag.
$C_8H_{15}O_2$ Ag. Ber.: Ag 43.00. Gef.: Ag 43.11; Ag 43.33.

Methylester der 1,2-Dimethylkapronsäure, Siedepunkt, 175—176°.

0.1207 g Sbst.: 0.3014 g CO_2; 0.1233 g H_2O;
0.1326 g Sbst.: 0.3303 g CO_2; 0.1335 g H_2O.
$C_9H_{18}O_2$. Ber.: C 68.35; H 11.39.
Gef.: C 68.10; H 11.43;
» : C 67.94; H 11.27.

Das Chloranhydrid der Säure, Siedepunkt 80°, 25 mm. Das Säureamid stellt Blättchen von dem Schmelzpunkt 121° dar.

0.1162 g Sbst.: 10.27 cm³ N (20°, 753.3 mm);
0.0903 g Sbst.: 7.78 cm³ N (17°, 743 mm).
$C_8H_{17}ON$. Ber.: N 9.79, Gef.: N 9.97; N 9.71.

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Akademie der Wissenschaften der UdSSR.
Moskau.

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CHEMISTRY

ON NEW COLOR REACTIONS BETWEEN TETRAVALENT CERIUM AND SALTS OF AROMATIC ACIDS

By F. M. ŠEMIAKIN and A. N. BELOKON'

(Communicated by N. S. Kurnakov, Member of the Academy, 17. XII. 1937)

This report gives an account of the results of an investigation of the reactions between cerium, thorium, praseodymium, neodymium, titanium, zirconium and salts of benzoic, salicylic, cinnamic, mandelic, picric, phenylacetic, quinic, naphthoic, β-naphthosulfonic, phtalic and anthranilic acids, and also with phenylalanine and phthalimide.

The reactions were studied in neutral, acid and ammonium solutions.

Of all the organic compounds tried by us, the following gave color reactions with tetravalent cerium: anthranilic, naphthoic and salicylic acids and also phenylalanine (Table).

Reagent	Nature of reaction	Reaction in hydrochloric acid solution	Reaction in ammonia solution
Ammonium naphthoate	Dark, lilac-black, voluminous, compact precipitate	Insoluble even in strong acid solution	Under the action of ammonia the black precipitate turns flaky and reddish-brown
Ammonium anthranilate	Abundant dark redbrown precipitate	Insoluble even in strong acid solution	Under the action of ammonia the precipitate turns chocolate in color
Phenylalanine	Dark blue-green precipitate	Dissolves, giving a dark green solution	Ammonia precipitates out a red-brown flaky precipitate and the solution becomes colorless
Ammonium salicylate	Dark, almost black precipitate	Insoluble, even in strong hydrochloric acid solution	Brownish-grey precipitate

The other compounds of the number mentioned above do not give characteristic color precipitates nor colors with tetravalent cerium, not counting the yellow precipitates with benzoate and picrate.

Salts of trivalent cerium, thorium, praseodymium, neodymium and zirconium do not give colored precipitates with salts of naphthoic, salicylic and anthranilic acids and phenylalanine. Part of these compounds has already been described in literature (1).

When trivalent cerium reacts with ammonium salicylate in ammonium solution, according to our experiments there appears at first a white precipitate which very slowly takes on various shades of pinkish red. If the same reaction is performed with tetravalent cerium, then in weak acid solution there appears an almost black precipitate which changes into a brownish-gray precipitate under the action of ammonia. Thus salicylate has a different effect on tetravalent cerium than on trivalent cerium.

Titanium and salicylic acid, as is known, give yellow precipitates in neutral and slightly acid solution (2), and trivalent cerium gives a red-brown precipitate in ammonium solution (3).

In connection with this it is of interest to note the analogy in the action of hydrogen peroxide and gallic acid on titanium and trivalent cerium (4). It must be stated that naphthoic, anthranilic and salicylic acids and phenylalanine have a certain resemblance in the arrangement of the reactive groups in their molecules. It is possible that their reaction with cerium has a specific connection with these common features in the structure of their molecules. These reactions are probably due to the high oxidation potential of tetravalent cerium. In connection with this it is worthy to note the color reactions between cerium and certain alkaloids, which were discovered by one of us not long ago (5).

We investigated the possibility of using the reactions of tetravalent cerium, described above, in drop analysis and colorimetry. Our experiments have shown that although the reactions described above give quite characteristic stains and colorings on paper, their sensitivity is not very great, about 10^{-3}—10^{-4} *N* cerium.

Conclusions

1. New color reactions have been described between tetravalent cerium and naphthoic, anthranilic and salicylic acids and phenylalanine.

2. Trivalent cerium, thorium, titanium, zirconium, praseodymium and neobium do not give color reactions with these substances.

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CHEMISTRY

NEW DROP REACTIONS OF VANADATES AND MOLYBDATES WITH α-NITROSO-β-NAPHTHOL

By F. M. ŠEMIAKIN and A. N. BELOKON

(Comminicated by M. A. Iljinsky, Honorary Member of the Academy, 11. XII. 1937)

As a reagent on cobalt, α-nitroso-β-naphthol was first applied by M. A. Iljinsky (1).

Later many authors investigated the possibilities of applying α-nitroso-β-naphthol in analysis. A summary of the results obtained was given by Weber (2). More literature on the subject may be found in the works of H. Beilstein (3), Rüdisule (4), Mellor (5), etc.

Besides cobalt, α-nitroso-β-naphthol has been suggested for precipitating iron and copper. Its reactions with palladium (6), trivalent vanadium (7), zirconium, molybdenum and titanium (8) are also known.

In this paper we present the results of an investigation of the reactions of α-nitroso-β-naphthol with vanadates and molybdates. This investigation has shown that the reaction of α-nitroso-β-naphthol with the anions indicated are quite characteristic and are apparently worthy of serious attention. The results attained by us are given below.

To promote these reactions α-nitroso-β-naphthol was dissolved either in 50 per cent acetic acid or in 96° alcohol. Solution in alcohol is more convenient, since a more complete deposit of vanadates and molybdates is observed in this case. We used saturated solutions of α-nitroso-β-naphthol.

The Reaction for Vanadium. An alcohol solution of α-nitroso-β-naphthol precipitates vanadium out of ammonium vanadate in acid as well as in alkaline solution. In alkaline solution pentavalent vanadium and α-nitroso-β-naphthol at first give a green color (sol). This sol quickly throws down a dark green precipitate.

The precipitation of vanadium out of the same solution, slightly acidified with hydrochloric acid (1 ml *n*/1 hydrochloric acid to 2—3 ml of vanadate), proceeds otherwise. In this case a voluminous brown-red precipitate is formed. In neutral solution no precipitate appears.

An acetate solution of α-nitroso-β-naphthol gives a brown-red precipitate of vanadium even in the absence of hydrochloric acid.

This brown-red precipitate of vanadium and α-nitroso-β-naphthol dissolves in concentrated mineral acids. In nitric acid it dissolves violently. Sulphuric acid dissolves the precipitate forming a dark red-brown solution. Hydrochloric acid promotes precipitation. A large excess of hydrochloric acid incompletely dissolves it. A solution of potassium hydroxide dissolves the precipitate forming a transparent greenish-yellow solution.

When great quantities of precipitate are dissolved the solution has

a dark red color. If hydrochloric acid is added in excess to the alkaline solution there again appears a precipitate of the original color.

The precipitation of vanadium with α-nitroso-β-naphthol is almost quantitative. After precipitation with α-nitroso-β-naphthol the filtrate was tested for vanadium with hydrogen peroxide and metallic zinc.

An acetate solution of α-nitroso-β-naphthol does not precipitate vanadium from vanadate solutions as fully as an alcohol solution. The sensitivity of the reaction between α-nitroso-β-naphthol and ammonium vanadate in hydrochloric acid solution is of the order of $5 \cdot 10^{-6}$ g vanadium in 1 ml.

This reaction may be conveniently carried on as a drop reaction. This is done in the following manner: by means of a capillary tube a drop of vanadate solution is deposited on filter paper, then a drop of tincture of α-nitroso-β-naphthol and finally a drop of hydrochloric acid in *n*/1 solution is added. The formation of a brown ring shows the presence of vanadium. The lowest concentration limit for the drop reaction is of the order of $1 \cdot 10^{-5}$ g vanadium in 1 ml of solution.

The Reaction for Molybdenum. The reaction for molybdenum in alkaline solution has been described by F. Steidler [8]. An acid solution of ammonium molybdate reacts with alcoholic as with acetate solutions of α-nitroso-β-naphthol, forming a voluminous velvety red precipitate. The action of acids and bases on this precipitate is analogous to their action on the vanadium precipitate.

The precipitate of α-nitroso-β-naphthol and molybdenum coagulates and filters very poorly. The precipitation of molybdenum is not quantitative.

The sensitivity of the reaction of α-nitroso-β-naphthol with molybdenum is of the order of $3 \cdot 10^{-4}$ g molybdenum in 1 ml of solution.

This reaction may also be brought about after a drop method. This is done in the following manner: by means of a capillary tube a drop of molybdate solution is deposited on filter paper, then a drop of tincture of α-nitroso-β-naphthol, and finally a drop of hydrochloric acid in *n*/1 solution is added. When this dries four rings appear: the outer one is sky-blue; the next is yellow, and is separated from the inner orange ring by a thin ring of lilac color. The outer sky-blue ring may not appear when there is an excess of α-nitroso-β-naphthol. The sensitivity of the drop reaction is in the vicinity of $1 \cdot 10^{-4}$ g molybdenum in 1 ml of solution.

Practically α-nitroso-β-naphthol does not precipitate the following ions: calcium (2), strontium (2), barium (2), magnesium (2), aluminium (2), nickel (2), chromium (3), manganese (2), zinc (2), cadmium (2), mercury (2), tin (2). With tincture of α-nitroso-β-naphthol the tungstate ion in neutral solution gives a yellow precipitate, which quickly dissolves when hydrochloric acid is added. The arsenate ion (AsO_3''') does not give a precipitate with tincture of α-nitroso-β-naphthol in neutral solution. When acidified with hydrochloric acid a little orange-red precipitate appears.

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GEOCHEMISTRY

ON THE BEHAVIOUR OF MONTMORILLONITE AT HIGH TEMPERATURES

By D. S. BELIANKIN, Corresponding Member of the Academy, and V. P. IVANOVA

The last investigations show montmorillonite similarly to kaolin to make a sufficient component of many clays, this mineral representing the absorbing qualities of the latter (bentonites, floridine clays). Therefore, the chemical and physical properties of this mineral which are not yet adequately studied are of great both theoretical and practical importance. The present paper in this respect reports some new data obtained in the course of the thermal analysis of several Soviet montmorillonite clays.

The first specially thermal investigation of montmorillonite was accomplished, as is known, by A. Le-Chatelier ([2]) just 50 years ago (1887). On the heating curve of this mineral from Montmorillon he stated three separate endothermal stops: at 150, 770 and 950°. Recently the similar curve by means of a more perfect differential method was obtained by Orcel and Caillère ([3]) (1933). This latter curve has stops at 150, 720 and 820°. Thus we see that the last stop is more than 100° lower than in the first case. During these years the thermal analysis of montmorillonites from several Soviet deposits were also made by the authors of the present work ([6]). The individual stops obtained for this mineral are represented in the table.

Temperatures of Endothermal Stops on the Heating Curves of the Soviet Montmorillonites

No	Individual name	Deposit	Temperatures in °C		
			1st stop	2nd stop	3rd stop
1	Askanite, sample № 1	Askana, Georgia	—135	612—640	805—850
2	Askanite, sample № 2	» »	—182	630—667	820—842
3	Bentonite clay	Canja, Azerbaijan	—127	600—635	790—818
4	Kil	Kurtzy, Krimea	—130	—640	779—826
5	Beidellite	Kobuleti, Georgia	50—110	610—643	755—826
		Aver	50—137	613—649	790—832

In comparison with Orcel's and Caillère's results we see that the temperatures of the second stop are considerably lowered, about 100°, on all

Soviet montmorillonites curves without exception. In the temperature columns of the table the figures before and after the dash refer to the beginning and the maximum development of the stop respectively (compare the heating curve of askanite № 1 in Fig. 1).

The physico-chemical interpretation of all three curves has been of a great scientific importance. Orcel and Caillère explained the first stop by the separation from the mineral of the adsorbed mechanically bound water (15—20%), and the second and third stops, by the yielding of the constituent water (5—8%). Hofmann, Endell and Wilm [1] (1934) considering the X-ray diagrams of the hydrous montmonrillonite, on one part, and the products of its dehydration at different temperatures, on the other, offered a similar explanation. The adsorbed water (n—1 molecules of H_2O), after these authors, occupies the inter-packet position in the crystalline lattice of the mineral, while the constituent water

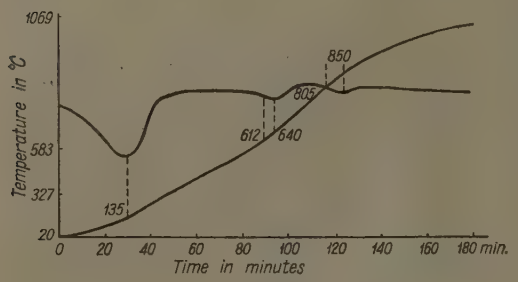

Fig. 1.

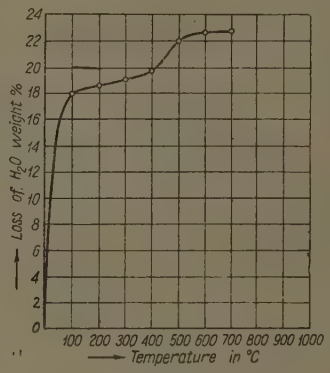

Fig. 2.

(1 molecule, corresponding to 5 weight per cent of H_2O) is introduced into the very packets; after the separation of the inter-packet water montmorillonite attains the pyrophyllite structure.

We could not of course be fully satisfied by these explanations since we could not see why the separation of the constituent water from montmorillonite was accompanied not by one but by two stops.

In order to find some way for an adequate explanation we consider first of all the experiments of montmorillonite dehydration by heating to various temperatures. Such experiments were carried out by different authors, the obtained results being in good agreement. The dehydration curve represented in Fig. 2 was obtained by the authors of this paper for the montmorillonite of Askana №1, having 23% of H_2O. In the range of this curve one can discriminate four separate intervals: from 0 to 100°, from 100 to 400°, from 400 to 500° and from 500 to 700° when 18, 2, 2 and 1% of H_2O are yielded respectively. Thus the first portion of the curve shows the separation from mineral of the whole quantity of adsorbed water; as to the constituent water, it begins to be yielded in small quantity already immediately after 100°. This process rather slow in the beginning shows a sudden jump-like acceleration at 400° and then after 500° its development is more and more slow. The full dehydration of the substance is reached at 700°. If now the curves of Fig. 1 and Fig. 2 are compared one sees the correspondance between two steep portions of the second curve (0—100° and 400—500°) and between two first stops on the first curve (50—133° and 618—649°), while for the third stop on the first curve we find no corresponding part on the second curve.

That this is really the case, i. e. that the third endothermal stop has no relation to the dehydration of montmorillonite substance, but corresponds to some additional thermal process in already dehydrated montmorillonite, was further on proved by the authors of this work by means of special experiments. In these experiments heating curves of askanite after its preliminary dehydration at 400, 450, 500 and 550°, were obtained.

The first two cases showed, as it could be expected, the disappearance on the heating curves only of the first stop; in the third case too there was weakened the second stop, which disappeared entirely in the fourth case; the third stop preserved its intensity constant in all the cases.

Having proved by these experiments that the third endothermal stop of montmorillonite is independent of the dehydration process we proceeded to our further work with pyrophyllite. As we have mentioned above, Hofmann, Endell and Wilm indentified the anhydrous form of this mineral with the structure of dehydrated montmorillonite. The especially obtained heating curve of classical Berezov pyrophyllite showed only one endothermal stop at 630—680° which as in the case of montmorillonite undoubtedly corresponds to the separation from the mineral of the constituent water. It is easy to understand the absence of the first montmorillonite stop on this curve since, as is known, the absorbed water is not present in pyrophyllite. But it is characteristic that we also do not find on it the third stop of montmorillonite. This fact contradicts the conclusion of Hofmann, Endell and Wilm as to the identity of crystalline structure of the dehydrated pyrophyllite with that of dehydrated montmorillonite. Concerning the dehydrated pyrophyllite we have now the X-ray data recently published by Tillo and Schünemann (4) (1937), which show that it preserves its characteristic crystalline lattice not only during the dehyration process but even at the further heating of the dehyrated mineral up to the temperatures above 1000°. Only above 1100° this lattice is entirely destroyed, minerals mullite and crystobalite being formed as the resulting products of disintegration.

By request of the authors, G. A. Kovalev obtained the X-ray diagrams after Debay's method of the four following preparations of Askana montmorillonite: 1) natural material, 2) the same material but previously heated with the same rate as in the case of the heating curves up to 700° (the temperature intermediate between the second and third stops), and then cooled in the air, 3) the same askanite heated as in the previous case but up to 900° (point lying above the third stop), and also cooled in the air, and 4) askanite heated up to 1100° with the same posterior air cooling. The results are as follows.

Natural askanite gave a typical X-ray diagram of montmorillonite similar to those obtained by Hendriks-Fry, Kerry and Nagelschmidt. The X-ray diagram of askanite previously heated to 700° preserved essentially all the lines of initial substance, their intensity being only a little smaller. The X-ray diagram of the preparation previously heated to 900° shows the entire disappearance of the lines of initial askanite; instead of them there appeared 4 new lines belonging to the new crystalline substance. The X-ray diagram of the preparation deviously heated to 1100° reveals at last a very intense development of these 4 lines with addition of many new lines which are absent in all the previous diagrams, as well as in X-ray diagram of pyrophyllite given for comparison.

These X-ray data are naturally explained as follows. During the process of its dehyration montmorillonite, similarly to pyrophyllite, preserves its fundamental crystalline structure, but during the further heating this structure is quickly destroyed resulting in the appearance of the third endothermal stop on the heating curve. The beginning parallel development of the

new crystalline phase at the expense of the disintegration products is, on the contrary, so slow that it does not produce any significant effect upon the heating curve.

The microscopic study which we have adjoined to the X-ray investigation shows that after the second endothermal stop our mineral still preserves its birefringence and the general appearance of montmorillonite. But after the third stop the preparation contains, together with the residuary montmorillonite of a reddish brown hue, also a new phase—an isotropic mineral of yellowish colour with $n=1.535\pm0.005$. The quantity of this new phase increases with the further growth of temperature and, at last (1100° of the heating curve) it replaces entirely the initial birefringent montmorillonite.

The last question we had to solve was: what is in mineralogical nature of the isotropic phase that appears during the heating of montmorillonite and that does not form in the case of pyrophyllite.

If we inquire into the chemical composition of these two minerals we see that montmorillonite except Al_2O_3 and SiO_2 contains also the oxides of RO type with the prevalence among them of MgO. This difference suggests the cordierite composition as the more probable for our new isotropic formation. This is more or less supported by its refraction index but is opposed by the absence of birefringence, by its optical isotropy.

The special X-ray investigation made by G. A. Kovalev shows that our isotropic product has no relation to cordierite but represents quite a peculiar original mineralogic phase. But its nature still remains unknown. The general points of montmorillonite heating curve interpretation are, therefore, as follows:

1. The first endothermal stop shows the separation of the adsorbed water.
2. The second endothermal stop corresponds to the separation of the constituent water.
3. The third stop reveals the destruction of lattice, the montmorillonite becoming amorphous.
4. The further even run of the curve corresponds to a gradual crystallization of a new mineral phase from the amorphous substance and to a gradual growth of its crystals.

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BIOGEOCHEMISTRY

BIOGEOCHEMICAL PROVINCES AND ENDEMIES

By A. P. VINOGRADOV

(Communicated by W. I. Vernadsky, Member of the Academy, 17. XII. 1937)

1. The changes in the character of soils affect in some way or other the vegetable cover, which in turn affects the animal population.

Here we shall discuss only one particular case, viz. the change of the chemical elementary composition of the medium, i. e., of soils, water and air.

As is known, these changes, occurring for instance in soils, being due to various causes, not only exercise their influence upon the crop but change the qualitative composition of the flora as well. Thus, for instance, the flora of calcareous soils and that of sandy (siliceous) soils are widely different. Connected with the former is the so-called calcium flora; with the latter, a numerous vegetation of cereals, etc., rich in silica. Connected with salines and saline soils is a vast halophyte flora, represented by many obligate plants—halophytes. The diversity of the chemical composition of soils of various types is not confined to a higher content of NaCl, $CaCO_3$, SiO_2, etc. In some soils, especially those located near some decomposing mountain massifs and ore occurrences, in regions of volcanic activity, etc., some minerals are frequently present in a disseminated form and in greater amounts than those usually present in soils; such are, for instance: serpentine, datolite, tourmaline, fluorite, apatite, boron minerals, compounds of nickel, copper, zinc., etc. Floras are known settling primarily on some similar soils; as early as the last century attempts were made to use the presence of these floras as an indicator in prospecting for ores; such are, e. g., the galmey flora (zinc), *Lonicerae* and other plants in the areas of gold placers, etc.

On serpentine soils, on galmey soils, on those with a high content of calcium, aluminium, chlorine, etc., among the vegetable cover appear varieties typical of these soils.

A change in the content of H^*, iron, manganese, calcium, sodium chloride and many other chemical elements and their compounds in the water of various land basin is immediately manifested in the characteristic changes appearing in the plankton and in other organisms. However, the influence of a certain excessive or, on the contrary, deficient content of individual chemical elements in the medium upon the flora and fauna is of a more general nature.

2. In different countries exist under different names numerous endemic diseases of various plants and animals, the cause of which is now explained by the geochemical history of the country, and the composition of soils.

Thus, for instance, the endemy of «brittle bones», that especially affects herbivorous animals, has been known for a long time and still exists at definite places in all countries. The cause of this endemy is a direct lack of calcium and phosphorus in the most diverse soils and vegetable cover, producing osteoporosis, «rickets» and other diseases of animals. From the lack of calcium in a given geochemical region or province suffers the whole animal population. Thus, for instance, among fresh-water mollusks, forms with thin valves appear—var. *temor*, etc. The phosphorus starvation is frequently experienced by the vegetation of steppes, savannahs, etc.

Such is the case not only with calcium but also with other alkaline—earth elements (strontium, barium, magnesium, beryllium); when the usual ratios of Ca/Sr, Ca/Mg, Ca/Be in a medium are disturbed, or when they are in a small excess, peculiar strontium, beryllium, magnesium and other «rickets» appear; now the strontium «rickets» has been obtained experimentally, the beryllium discovered with animals from soils containing beryl (Michigan, U.S.A., etc.). Thus all of the akaline earth elements under certain conditions produce diseases of the bone system*. The changes produced by the presence of serpentine in soils (appearance of «serpentines varieties» among the flora) have been accounted for by an excess of magnesium, the calcium content being low. Serpentines (and serpentine soils having a wide distribution**) frequently contain higher percentages of chromium and nickel. W. Robinson was the first to suggest that it is chromium that affects the flora from serpentines.

Another widely spread endemy is the endemic goitre affecting people and animals, which has nidi in all countries, chiefly in high-mountainous localities or in regions with a moory landscape. The chief reason is a lack of iodine in the surrounding medium high-mountainous areas due to the geochemical history of iodine. The map showing the endemic distribution of endemic goitre, coincides with the map of the deficiency of iodine in these areas. The introduction of iodine into the food (iodine-prophylaxy) prevents the development of endemic goitre. As is well known iodine enters in the hormone of the thyroid gland.

For some hundreds of years in Iceland an animal disease has been known, called «gaddur». It becomes stronger after the eruptions of the local volcanoes. In Italy, near Naples, e. g. in Pozzuoli (a volcanic region) cases were observed when the aborigines had a mottled enamel of the teeth, called «denti neri». This disease was then described from the United States, Africa and other countries (under the name of «mottled enamel», «dientes veteados», «Darmous»). All the animal population of a given region suffered from the disease. The fishes in the endemic regions had mottled enamel, etc. The cause was found to lie in a small excess of fluorine in drinking water, soil and plants. As the source of fluorine served volcanic eruptions***, tuffcone and fluorine-apatite, etc., both in accumulations and in a disseminated form in the soils of there localities. Especially strong is this endemy in North Africa near the renowned deposits of phosphorites (Tunis, Algiers etc.).

In some localities of Nebraska, South Dakota and Wyoming for a long time an endemy has existed, producing a peculiar chlorosis of vegetation and a special «alkali disease» of horses, cattle and people. Only in 1931 it was established (K. Frank, H. Knight) that the cause of this endemy lay in

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* With a lack of calcium and the feeding of animals on cereals, rickets appears known under the name of «siliceous rickets».

** In the Soviet Union in the Urals-Khalilovo and other places.

*** Volcanoes bring hundreds and thousands of tons of HF to the air annually.

a higher selenium content in the soils of these regions rich in pyrites. The plants reacted to selenium in a different way. Selenium partly replaced sulphur in albumens, forming poisonous albumens.

It has been at present established that the absence of boron in the medium prevents the growth of a number of plants. In beet appears the so-called heart-rot; in leguminous plants the nodules are not developed, etc. On the contrary, on soils relatively rich in boron (more frequently in arid regions) in many plants another peculiar disease appears (e. g. in citrus trees).

An excess or deficiency of elements of the group of iron also causes analogous diseases. A deficiency of iron in the vegetable cover is shown by the appearance of a peculiar chlorosis. A most diverse anemia of animals, known under the name of «Bush sickness» (in New Zealand) and «pining» (in Scotland), and many others are due to a deficiency of iron in the food. Extremely interesting are the facts observed by New Zealand investigators H. Askew and others, namely, that the bush sickness of animals may be cured by introducing into their fodder some varieties of limonites and other iron ores containing, as it has been shown, cobalt. It seems that here a disease has been established connected with a deficiency of cobalt.

Diseases of plants and animals are known connected with a deficiency of manganese, zinc and other heavy metals.

Particular interest is presented by a disease of plants known under the name «reclamation disease», «Heidemorkrankheit», etc., spread over vast areas of moory soil (gained from the sea) in Holland, North Germany, Danemark, and also in Poland, in the White Russian SSR, and in the United States. An introduction of copper to the soil cures the plants. These soils are poor in copper. With a lack of copper in the food some diseases of animals and people are connected*.

Finally, the presence in the soils of varying amounts of a number of other heavy metals—vanadium, molybdenum, lead, etc., undoubtedly exerts an influence upon the organism, particularly, on the population of soils (edaphone). It is highly probable that in some cases, e. g. among lower organisms this leads to the appearance of races, stable with respect to certain concentrations of definite chemical elements, while in others their presence may bring about a natural sterilization of soils with regard to definite microorganisms.

There is no doubt that we still ignore many things in this connection, although we may guess something (e. g. concerning the role of arsenic, sulphur, bromine, etc.).

3. Thus, the d e f i c i e n c y o r e x c e s s** of some given chemical element in the environment—in soils of different types, in soil and subsoil solutions, waters, plants, etc., as compared with their usual clark***, produces peculiar changes in the flora and fauna of a given region or b i o g e o c h e m i c a l p r o v i n c e. The organisms—flora and fauna—meeting provinces of such a kind either 1) perish, or 2) undergo a morphological

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* The copper exchange in higher animals is connected with a porphyrine exchange. In the regions with a low copper content this brings forth the development of organisms having hemocyanin in their blood, as for instance Mollusca, etc.

** The deficiency or excess of some chemical element or other may be also a relative one due to 1) the influence of some amounts of other chemical elements with closely similar chemical properties, e. g. Ca in the presence of Sr, Mg; Fe in the presence of Mn, etc...and 2) the different availability for different organisms of various combination of chemical elements.

*** Clark—average content (percentage) of a certain chemical element in rocks, soils, etc. This name was proposed by A. Fersman, Member of the Academy of Sciences of the USSR.

change, forming varieties, morphs, etc., or fall ill; 3) or are preserved without apparent external changes. In this latter case the appearance is probable of a certain physiological resistance to certain concentrations of chemical elements, as well as the appearance of races among them that are resistant to nickel, copper, zinc, etc.

Different endemic diseases—b i o g e o c h e m i c a l e n d e m i e s, arising among organisms, due to the deficiency or excess of some chemical elements, are characterized by 1) occupying a definite area—a biogeochemical province ranging in size from a spot to many millions hectares; 2) often embracing simultaneously the vegetable cover and the animal population; 3) striking chiefly the young population; 4) being due to a deficiency or excess of some chemical element in the medium. The last fact distinguishes these endemies and other changes of flora and fauna from geographical changes and geographical endemies*.

In the course of time through the existing and newly formed biogeochemical provinces various floras and faunas connected with them are spreading. A biogeochemical province is a stage on their way. Therefore a discussion and study of the influence of these provinces on the evolution of organisms, due to their influence upon selection, is a problem of the greatest interest.

The present paper is only a very brief summary of the essential points of the problem, which is discussed by the author at greater length in a paper prepared for publication.

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* An example of geographical endemies is furnished by the spreading of different kinds of fevers caused by stinging and biting insects, etc.

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MINERALOGY

ON THE TERRESTRIAL ALUMOPHOSPHOROUS AND ALUMOSULPHUROUS ANALOGUES OF KAOLINIC ALUMOSILICATES

By W. I. VERNADSKY, Member of the Academy

1. In the chemistry of natural silicates, some facts have lately come to light that would appear to have a strong bearing on the elucidation of the chemical structure of silicates and alumosilicates. These facts have been discovered by mineralogists by their own methods, not by chemists.

In that branch of chemistry, no current scientific work ought to be based on the existing theoretical chemical conceptions. Owing to geological sciences, and, to a certain extent, to technical, we are faced here, at nearly every step, with facts and phenomena, which, from the standpoint of the chemistry, are quite unexpected and run counter to customary chemical theories.

J. MacConnell's recent work (September 1937) ([1]) enables us to determine with greater precision the relations of kaolinic alumosilicates to alumosulphates (alum) and alumophosphates, whose similarity of chemical structure I pointed out in my lectures as early as in the last century, and also mentioned briefly but more definitely in recent years*.

MacConnell made a more detailed examination of the properties of wikleyite (from Crestmore, California), a contact mineral described A. Eacle and A. Rogers ([2]), rich in silicon and sulphur, which they classed, however, as an apatite, or complex phosphate **. Such an investigation was made by MacConnel on new material (from the same locality).

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* In my endeavours to bring out the similarities in the chemical behaviour of aluminium and silicon oxides in silicates, I was led, at the very beginning of my lectures at the Moscow University in 1891, to revise from that standpoint, not only the whole group of silicates, but also all the minerals containing aluminium oxide or its analogues (ferric oxide etc.), including those alumosulphates (alums) and alumophosphates that are the most widely distributed on Earth. This was a result of the admission that aluminium oxide is, within the parameters of terrestrial reactions, an acid anhydride. I had satisfied myself previously that such an interpretation of their chemistry was chemically admissible and did not give rise to any serious objections from the chemical point of view. Cf. W. Vernadsky, Bulletin de la Soc. d. Natur. de Moscou, 1891, p. 44, and my Introduction to the book of. B. Вернадский и С. Курбатов, Земные силикаты, алюмосиликаты и их аналоги, 25 (1937). As to alumosulphates and alumophosphates, I first mentioned this in print, if I am not mistaken, in the edition of a course of lectures I had delivered at the Sorbonne. Cf. W. Vernadsky, La géochimie, 173—174 (1924). In the Russian edition of the same book [«Очерки геохимии», 316 (1927)] I ponited out the theoretical probability of phosphorus and sulphur analogues (less silica) of the kaolin nucleus, and gave formulas. The same formulas were reproduced in the German edition of 1930 [«Geochemie», 315 (1930)].

** W. Ford [E. S. Dana, Textbook of Mineralogy, 706 (1932)], assigns it «tentatively» to the apatite group. In Doelter's Survey (C. Doelter, Handb. d. Mineral. Chemie, 111, I, 592 (1918)) H. Leitmeyer places it quite separately from the apatites, amongst the silicosulphocarbonophosphates.

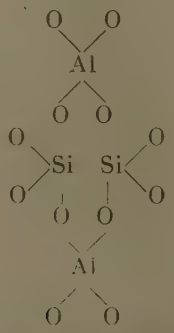
He confirmed and developed the ideas of Eacle and Rogers and proved by using new roentgenologic and crystallophysical methods unknown in 1914, that the space lattice of apatites extends to wikleyite as well as to ellastadite, a new associated mineral discovered by himself. The analyses were made by R. Ellastad.

According to the only analysis available, ellastadite contains: SO_3, 20.29%; SiO_2, 17.51%; P_2O_5, 3.06; CO_2, 0.61%, and fully coincides with fluorapatite $3Ca_3P_2O_7CaF_2$ (from Faraday Township, Ontario), with which MacConnel compared it in his measurements and roëntgenograms.

MacConnell has proved that P, Si and S form atomic tetrahedral complexes in fluorapatite and ellastadite, the atoms of P, C*, Si and S being in the centres of the tetrahedrons, while oxygen atoms are in the angles. These tetrahedral atomic complexes are fully analogous to those that build up the kaolin nucleus and are characteristic of kaolinic alumosilicates, where this complex is made up of (SiO_4) and (AlO_4) and their isomorphous analogues (TiO_4), (FeO_4), etc.

Starting from this structure of ellastadite, we must conclude that a number of chemical elements of so different a chemical nature as Si (tetravalent), S (hexavalent), or P (pentavalent), possibly also C (tetravalent), can, irrespective of their valency or isomorphism, replace one another in the solid crystalline matter of the planet, without the space lattices being disturbed.

When the acid anhydrides of this type $[SiO_4]$, $[AlO_4]$, $[SO_4]$, $[PO_4]$**, possibly $[CO_4]$, meet or are formed together, they give rise, in the conditions of terrestrial chemical reactions, to new and more involved ring complexes. The importance of one of these, the kaolin nucleus, Al_2SiO_7 or metanacrite,



and its very essential role in the matter of our planet, were pointed out by the writer *** a long time ago.

2. In 1891 after a critical survey of practically the whole of the available literature on the chemistry and mineralogy of silicates, accompanied by experimental work, the writer was led to assume the existence of a great chemical analogy between aluminium and silicon

* MacConnell admits the presence in ellastadite of atomic tetrahedral complexes of $[CO_4]$ substituting $[PO_4]$ and $[SiO_4]$. But in the apatite group, carbon atoms may (francolite) replace CaF_2 as $CaCO_3$. Cf. below.

** On the presence of tetraheders of $[PO_4]$ in the phosphates S. Hendrichs [S. Hendrichs, Amer. J. of Sc., **14**, 269 (1927)], [J. West, ZS. f. Krystall., **74**, 832 (1930)], S. Naray-Szabo [ib., 78, 596 (1930)], M. Memel (ib., 313).

*** Cf. my Introduction to the book B. Вернадский и С. Курбатов. Земные силикаты, алюмосиликаты и их аналоги (1937). Cf. W. Vernadsky, ZS. f. Krystall., **84**, (1933) (lit.), В. Вернадский, Очерки геологии (1934) (lit.).

oxides in the Earth's crust, of the existence of alumosilicic acids and their salts, and of the chemical complexity of natural silicates, not all of which are salts of silicic acids, as this was commonly thought in those days. These conceptions were briefly set forth by the writer in 1891—1901 (3). It may now be stated that the roentgenometric examination of crystals, first applied to silicates in 1927, has in the main confirmed my deductions.

At present we may go farther and assert as a fact that the «kaolin nucleus» is but a special case, but the most important one in terrestrial chemistry. A great number of minerals have in the main the same structure.

Starting from the nature of the chemical reactions taking place in our planet, we see that the kaolin nucleus, reduced to its simplest expression, consists of two oxygen-aluminium atomic tetrahedra, associated with two oxygen-silicon atomic tetrahedra, while alumosilicates that have a kaolin structure may contain, for one kaolin nucleus, as many as twelve, and possibly more, addititional oxygen silicon atomic tetrahedra. In the case of alumosilicic salts and hydrates of kaolinic structure, the silicon and aluminium atoms contained therein capture all the oxygen atoms of these compounds. The resulting honeycomb structure made up of atomic tetrahedra $2\ (AlO_4)2+n(SiO_4)$ corresponds to the structure of the kaolin nucleus*.

In the light of the facts ascertained by MacConnell, we may take a step further and assert that, under certain conditions, the silicon-oxigen atomic tetrahedra may be replaced in the kaolin nucleus, without any great disturbance of the crystal lattice, by similarly built atomic phosphorus-oxygen, sulphur-oxygen (possibly carbon-oxygen) tetrahedra. Such are improperly called basic and double aluminium salts, phosphates and sulphates, which, as far back as 1891, the writer classed as derivatives of alumophosphorous and alumosulphurous complexes. Such complexes form with water free acids analogous to clays. Their salts are to be found in such minerals as alunite, or minervite.

I shall briefly consider here these conceptions, which have found a considerable empirical support in the discovery of ellastadite.

They are in contradiction with the prevalent conceptions of chemists. Here, as in the case of many minerals, the existin gconceptions of inorganic chemistry ought to be thoroughly revised.

3. As stated by the writer in 1924 (4), both mineralogy and geochemistry contain serious evidence of the existence of alumophosphorous and alumosulphurous compounds, genetically associated with alumosilicates of kaolinic structure and formed from kaolin alumosilicates in the course of various surface or hydrothermal processes, as well as a consequence of post-volcanic phenomena, their formation being bound up with the replacement of silicic oxid by the oxides of phosphorus or sulphur. It is a matter for regret that these problems have so far been but imperfectly studied, and the chemical formulas of some of the principal minerals concerned lack precision. The natural process of the formation of alumosulphates and alumophosphates cannot essentially be reduced (at least without radical and far-reaching corrections) to chemical phenomena as set forth in treatises on inorganic chemistry, since the dynamics of natural chemical reactions differs from that underlying those reactions in chemistry, and what we deal here with are facts and parameters which chemistry does not take into account

* In the case of kaolin nuclei, this holds good for anhydrides, acids and salts, e. g. for feldspars. In the case of more complex bodies, e. g., sodalite, the nuclei capture only the most of the oxygen atoms.

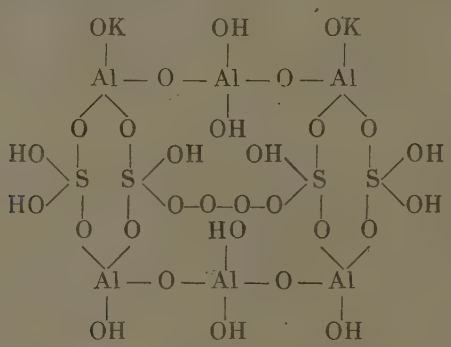
and which alter considerably its theoretical conceptions. First of all, this process is often bound up with biochemical phenomena, whose importance is, as rule, overlooked.

Moreover the process takes place simultaneously in the solid, liquid and gaseous phases. In the natural process of their genesis, a considerable portion of the solid of the initial kaolin alumosilicates does not pass into liquid or colloidal solution. The structure of the resulting alumosulphates or alumophosphates retains the material imprint of the kaolin nucleus. Some of the atoms of the kaolin nucleus may remain undisplaced during the reaction. The reactions resulting in the formation of natural minerals therefore bring to light, in a clear and forcible manner, the atomic structure of the alumophosphate or alumosulphate.

4. Two minerals, alunite and minervite, are mainly formed, in large quantities, in the course of the better studied natural processes of this kind. The empiric formula of alunite does not give rise to any doubt; that of minervite (the alumophosphate) is often incorrectly stated in reference books. I adopt the correction introduced into it by F. Zambonini ([5]).

As shown by analyses, both these potassium compounds, genetically associated with kaolin, and not infrequently with potassium alumosilicates, such as feldspars, micas or kaolin clays, show in their structure alumosulphurous or alumophosphorous nuclei, analogous to the kaolin nucleus, as well as atomic oxygen tetrahedra characteristic of apatites and ellastadite.

Here is the formula of alunite as deduced from the data of his analyses and based on the above considerations:



It corresponds to the following empirical formula:

$$K_2Al_6(OH)_{12}(SO_4)_4 \quad \text{or} \quad K_2SO_4Al_2(SO_4)_3 \cdot 2Al_2(OH)_6$$

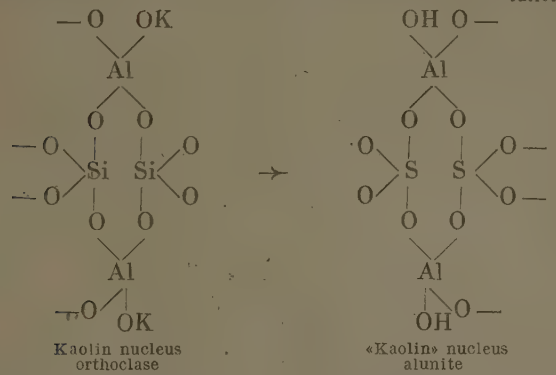
What is hypothetical in the above structure formula of alunite is the ringshaped, kaolin-like arrangement of the tetrahedral sulphur-containing oxygen complexes. In the formula of this mineral, the additional aluminium atoms, as shown by analysis, arrange themselves outside the kaolinic alumosulphurous nuclei. That two kaolin nuclei must exist becomes evident from the data of the analysis. What we see here is the same structure as is characteristic of the kaolinic alumosilicates: all the oxygen atoms of the alunite arrange themselves in the angles of the tetrahedra, which have aluminium or sulphur atoms in their centres. All the other atoms, those of hydrogen and potassium, are connected with the tetrahedra through oxygen. The additional atoms of aluminium should yield the same atomic

tetrahedra of aluminium [AlO_4] as those that are found outside the kaolinic nucleus*.

The genesis of alunite may to a certain extent be deduced from its formula.

In the first place, there are many pseudomorphs of alunite after orthoclase found in various localities, connected with the disappearance of SiO_2 and its replacement by SO_3 and H_2O. Ultimately, two thirds of the aluminium atoms remain in the «kaolinic nucleus» of alunite:

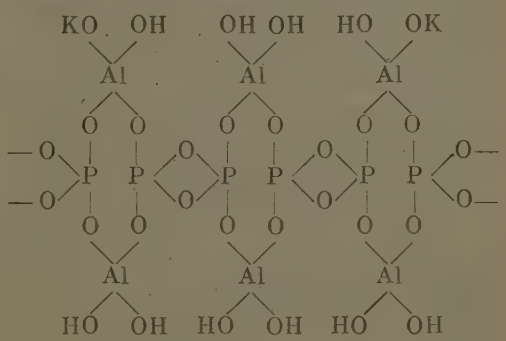
$$\underset{\text{(orthoclase)}}{3K_2Al_2Si_6O_{16}} + 6H_2O + 4SO_3 = \underset{\text{(alunite)}}{K_2Al_6(OH)_{12}(SO_4)_4} + \underset{\text{pass into aqueous solutions or colloids}}{2K_2SiO_3 + 4SiO_2}$$



One third of the aluminium atoms of orthoclase yield atomic tetrahedra of aluminium with two hydroxyl groups—the two alumosulphurous «kaolinic» nuclei being connected with them through oxygen.

Alunite is not infrequently formed in volcanic regions and sometimes occurs in deposits of considerable thickness. Its chemical association with feldspar is here quite obvious, but the process cannot yet be quite definitely accounted for. Its active agents are H_2SO_4 and SO_3 (6).

The same process may go on outside volcanic regions due to the action of sulphuric acid and possibly in the form of biogeochemical manifestations, i. e. in hydrothermal or surface processes, the new minerals formed being alunite, kaolin, boxite and silicic hydrates (6). These phenomena call for special attention and scientific investigation.



5. As to alumophosphates, the following formula has been derived for minervite, the chief amongst these minerals:

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* The metal a l u m i a n is mentioned whose formula in tetrahedra corresponds to $Al_2S_2O_{12}$, i. e. to a saturated «kaolin» alumosulphur nucleus. Cf. B. В е р н а д с к и й, Очерки геохимии, 4th edition, 111 (1934), Idem, Introduction to B. В е р н а д с к и й и С. К у р б а т о в, Земные силикаты..., 40 (1937). The formula of alumian cannot, however, be considered as proved. Synthesis is necessary, the more so as the artificial alumosulphates having their Al:S ratio equal to unity have been but imperfectly studied and possess some unusual properties. The $Fe_2S_2O_{12}$ complex is represented by a number of minerals, some of which are very common (amaranite, fibroferrite, etc.). Concerning alumian cf. C. H l a w a t c h. D o e l t e r's Handb. f. Mineral. Chemie, 4, 381 (1929).

It corresponds to the empirical formula for minervite as deduced from the analysis of E. Casoria (1904), corrected by F. Zambonini (1907). It provides for a slightly higher percentage of aluminium than found by Casoria, but the aluminium to phosphorus ratio, which is taken to be 1. has been proved beyond doubt by a number of Carnot's analyses. By doubling that formula I obtain: $H_4K_2Al_6(PO_4)$. $14H_2O$. I do not intoroduce into the structural formula any water separated below 180° and take into account only that separated at red heat. It is, of course, well known that the distribution of water cannot be inferred from a roentgenogram with due precision and without recourse to a hypothesis.

Similarly to the formula for alunite (§ 3), this minervite formula does not involve any hypothesis; it expresses the structure of the mineral in the form of three «kaolinic» ring complexes consisting, respectively, of two aluminium atomic tetrahedra and two phosphorus atomic tetrahedra. All the oxygens (7) of minervite are captured by them. The formula fully agrees with the analytical data and with the three «kaolinic» alumophosphorous nuclei. The atoms of potassium and hydrogen respectively combine with the atomic tetrahedra through oxygen. This is what is meant by the «kaolinic» nuclear structure. It appears to hold good for a vast majority of natural alumophosphates.

These formulas must of course, first of all, be verified by the roentgenometric method both for alunite and minervite.

As regards minervite, the main features of the process have been thoroughly and correctly outlined by A. Lacroix (8). Still, the process has so far been elucidated only in the first approximation. It is a biogeochemical process going on in the wide torrid climate zone extending over 17° on both sides of the equator. Minervite is formed from the kaolinic alumosilicates of massive and metamorphic rocks (feldspars, micas, etc.) through the action of solutions of phosphates mainly of ammonia and of alkaline metals, during the biochemical change of guano. Jointly with these, acid phosphates are formed. Many imperfectly studied pseudomorphs of minervite after kaolin alumosilicates are known. The process is accompanied by the separation of silica, partly as opals. In many instances, the kaolinic structure of the nucleus is preserved, the whole of the silica and part of the potassium being lost and the water absorbed. Three «molecules» of orthoclase yield one «molecule» of minervite. Some of the atoms of the kaolinic nucleus do not appear to be displaced during the process.

6. It may be inferred from the above by way of a general empirical deduction that the ability of some structures of terrestrial minerals to yield atomic oxygen tetrahedra (MO_4) does not depend on the chemical valency of the M atom contained in them and lying in the centre of the oxygen tetrahedron.

Trivalent aluminium, tetravalent carbons and silicon, pentavalent phosphorus, hexavalent sulphur, all form atomic tetrahedra of practically the same geometrical dimensions, all combine with four atoms of oxygen. These tetrahedra can replace one another in the structure of minerals without disturbing their homogeneity.

Chemical properties, however, manifest themselves to a certain extent. From the properties of the kaolinic nucleus it is known that the isomorphous series of M elements may be taken as a reliable indication of the presence of geometrically related oxygen tetrahedra with an M element in the centre.

The study of firmly established kaolinic nuclei of Al and Si warrants the assumption that the trivalent atoms of Fe, Mn, Ti, V, etc., may give rise

in terrestrial minerals to geometrically related atomic tetrahedra of $[FeO_4]$, $[MnO_4]$, $[CrO_4]$, etc. The same tetrahedra must be yielded by all of the 16 rare earth elements. For Si we have respectively the tetrahedra of $[TiO_4]$, $[CeO_4]$, $[SnO_4]$, etc. Thus, no less than 40 chemical elements can give rise, in the chemistry of our planet, to geometrically related atomic oxygen tetrahedra.

The isomorphous series have not been deduced from the chemical properties of the elements. They are an empirical generalization of the chemical composition of solid minerals in the parameters of the thermodynamic fields of the Earth's crust. A connection with valency no doubt exists, but it is a very complex one, and the existence of related tetrahedra does not follow from valency.

A still more striking manifestation of the independence, or complex dependence, of these phenomena on chemical forces is to be found in the power of atomic tetrahedra, closely related geometrically but differing widely as to their chemical properties, to give rise to complexes consisting of two different pairs of tetrahedra,—the kaolinic nuclei.

The tetrahedral structure, so stable in the chemistry of the Earth, must, therefore, be due to some other cause, either more potent or more closely associated with the properties of the planet, i. e. to a cause of planetary, rather than of a purely chemical nature.

7. One of the most striking features in the chemistry of the Earth is the power of atomic oxygen tetrahedra to form «kaolinic» nuclei.

As may be inferred from the structure of the apatite group, this power of atomic oxygen tetrahedra to replace one another «isomorphically» is due to their geometrical similarity, being entirely unconnected with the chemical properties of the atoms occupying a central position in the tetrahedron. The isomorphous series are no longer observable here. This is a property of atomic tetrahedra, not of M atoms. It would thus appear that also the complicated complexes of the atomic tetrahedron, the kaolinic complexes, are due to other forces than the chemical ones.

8. The instance of the silicoaluminic kaolin nucleus suffices to show that we deal here with a planetary phenomenon, due to forces more potent that the chemical forces of aluminium and silicon.

I have repeatedly pointed out the paramount importance, in the history of our planet, of the endothermic effect attending the formation of the kaolinic nucleus ([9]). A considerable portion of the heat of radioactive decomposition is already absorbed by it in the process of its formation in liquid magma. The kaolin nuclei found in the rocks derived from them occupy a prominent position in the mass of the lithosphere.

The causes of these phenomena are to be sought for in the properties of the Earth's crust, not in chemistry. Some indications of this are available.

9. The following chemical phenomena constitute the most striking features of the structure of the Earth's crust and of its chemistry:

In the first place, a strong predominance in it of oxygen, silicon, aluminium, and iron atoms (in all 75.5% of the total number of its atoms) ([10]); in the second place, a strong predominance of oxygen atoms in the volume of the lithosphere and in its space lattices [91.7% of the volume of the solid Earth's crust according to V. Goldschmidt ([11])]; in the third place, the extreme strength of the chemical structures of the lithosphere, the immobility of the vast majority of its component atoms

for milliards of years. Chemical reactions are an exception in the mass of the lithosphere.

The inference to be drawn from these general empirical deductions is that such chemical bodies ought to predominate in the lithosphere which 1) can capture and permanently retain the greatest possible number of oxygen atoms, and 2) can form the most durable space lattices, i. e. such as would contain the greatest possible number of atoms per unit of volume.

«Kaolinic» nuclei realize the first condition. The simple compounds corresponding to them such as «kaolinic» anhydrides, hydrates and salts, capture all the atoms of the compound without exception (and, in other more complex cases, an overwhelming majority of oxygen atoms). Other tetrahedra associated with the kaolinic nucleus, e. g. in alunite, capture the largest share. «Kaolinic» bodies may be regarded as structural forms admitting of the greatest possible capture of oxygen atoms in the solid matter of our planet.

Atomic tetrahedra realize the second condition, since this is a case of a geometrical distribution of points, which (whether atoms or any other disperse manifestations are involved) is the most stable and durable one and therefore capable of holding the greatest possible number thereof per unit of volume.

In reality, both these phenomena are due to the great length of time—viz. milliards of years—during which our planet has been under conditions similar to those under which it is now. A stable dynamic equilibrium has thus been formed, the free energy of the planet being at a minimum.

10. In concluding this notice I should like to call attention to two problems, which are based on the preceding considerations and are capable of experimental verification.

In the first place, are all the «kaolinic» nuclei endothermic compounds, which, in my opinion, would agree with the same principle of the minimum of free energy of the solid matter of our planet?

In the second place, are there adequate proofs of the existence of atomic tetrahedra with a carbon atom in the centre, or of the probable existence of their «kaolinic» nuclei? What are, for instance, the properties of $Al_2C_2O_7$? Such carbonic bodies must obviously be formed outside the biosphere conditions, and they must exhibit that chemical analogy between carbon and silicon which we fail to observe in the usual compounds of carbon either in our laboratories or in the chemistry of the Earth, since the living substance provided no room for the manifestation of such an analogy, although, chemists have been aware of this existence for some generations past and have been looking for some unequivocal manifestation of that analogy.

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MINERALOGY

ON OSMIRIDIUM. OSMIRIDIUM WITH A CUBIC CRYSTALLINE LATTICE

By O. E. ZVIAGINCEV

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Natural osmiridium occurs together with platinum in the Urals and in many other places deposit, of the USSR and in other countries, both apart and as admixture to gold. Osmiridium attracts our attention by its composition being not constant but ranging from 20 to 70 per cent of osmium.

In the preceding research the author has established together with B. K. Brunovskji (1) that osmiridium with an osmium content of 35—70 per cent represents solid solutions of iridium (and other metals) in osmium, the crystalline hexagonal lattice of osmium being retained.

The analyses of St. Clair Deville (2) show the existence of osmiridium with a content of osmium lower than 35 per cent. The crystalline lattice of such specimens of osmiridium has not yet been studied. The author has studied a mineral closely related to osmiridium, named auric osmiridium or aurosmirid, containing:

Iridium	51.7%
Osmium	25.5%
Ruthenium	3.5%
Gold	19.3%

Aurosmirid possesses the cubic lattice of iridium. Thus the possibility has been proved of the existence of solid solutions of osmium and other precious metals in iridium with a retention of the cubic lattice.

The author aimed to increase the number of the studied minerals, presenting solid solutions, where iridium with its lattice of a cube with centred faces is the solvent. With this aim in view he undertook the investigation of a mineral he had extracted from the «residues» of the affinage of Uralic platinum.

The crude platinum being dissolved there remains an «insoluble residue», from which the coarse-grained (0.3 mm and more) portion has been sifted off. The author has examined such coarse siftings at the State Refinery in Sverdlovsk during several months of 1936. He selected those particles which did not flatten on the anvil when hammered, as the particles of platinum do, and did not split along the cleavage unlike osmiridium. These particles are of a light-silvery color and have a metallic lustre; they are very hard (scratch agate) and insoluble in boiling aqua regia. When

slightly hammered the particles of the mineral become somewhat flattened; when vigorously hammered they split showing a conchoidal fracture.

The author succeeded in selecting some grains out of several dozen grains of coarse siftings; one of these was cleaved and a part of it, weighing 0.0442 g was subjected to chemical analysis after a previously described method (4). The results of the analysis were as follows:

Sample	0.0442 g	
Iridium	0.0289 g,	65.4%
Osmium	0.0138 g,	31.2%
Rhodium	0.0008 g,	1.8%
Total		99.4%

Gold, palladium, platinum, ruthenium, iron and copper have not been discovered.

Another part of the mineral grain remaining from the chemical analysis was broken by crushing between two hardened steel plates and subjected to X-ray examination. The analysis has been carried out by B. R. Brunovskij in the X-ray laboratory of the Lomonossov Institute of the Academy of Sciences of the USSR, after the method of Straumanis. In the table below the data of the X-ray photograph interpretation is given showing that the mineral has a cubic face-centred structure.

Cu - rays; $2\rho = 0.027$ cm; exposition — 2 hours, $2\pi R = 21.082$

$2r$	$2r_0$	ϑ	$\sin\vartheta$	Indices	$\frac{\sin^2\vartheta}{\Sigma\ in^2}$	a
4.34	—	—	—	—	—	—
4.848	4.820	20°.58	0.1235_5	111	0.04116	3.7936
5.62	5.611	23°.96	0.1649	200	0.04120	3.7918
8.20	8.184	34°.93	0.3278	220	0.04097	3.8023
9.87	9.870	42°.12	0.4498	113	0.04088	3.8065
10.43	10.403	44°.40	0.4895	222	0.04079	3.8108
6.69	6.707	61°.37	0.7704	133	0.04055	3.8220
6.02	6.039	64°.22	0.8109	240	0.04055	3.8220

The value of the crystalline lattice parameter of the mineral $a = 3.8415 \pm \pm 0.001$ for 90° has been obtained by interpolation.

According to Owen and Iball (5) and Owen and Yates (6) $a = 3.8312 \pm \pm 0.0005$ for pure iridium.

Thus the lattice of the mineral under examination is qualitatively similar to the cubic face-centred lattice of iridium and its parameter is closely approaching to that of iridium. The physico-chemical nature of this osmiridium is clear: it is a solid solution of osmium in iridium.

The present investigation establishes the boundary of two solid solutions in the system of the osmium—iridium alloys: it lies between 35 and 31% osmium. Alloys with an osmium content above this limit have the hexagonal crystalline lattice of osmium; those below this limit, the cubic face-centred lattice of iridium.

According to the classification of V. I. Vernadsky (7) osmiridium shoud be subdivided into two varieties: nevanskite and syssertskite. In the former iridium is prevalent; in the latter, osmium. The boundary of these two varieties lies in the region of a 50 per cent content. In the author's opinion it would be more expedient to refer the boundary between syssertskite and nevyanskite to the region of 35% of osmium content and to consider as ne-

vyanskite the cubic varieties of osmiridium, and as syssertskite the hexagonal ones. However, lest the already established classification should be altered, it seems better to subdivide the system of natural osmium—iridium alloys into three parts:

1) from 0 to 35% osmium—minerals of the native iridium group,
2) from 35 to 50% osmium—nevyanskite,
3) from 50 to 70% osmium—syssertskite.

(No minerals have been found with an osmium content above 70%).

To the first group there refer native iridium (without osmium) containing usually an admixture of platinum, the above-mentioned aurosmirid and the iridium, described here, containing osmium. All these have a cubic face-centred lattice.

The second and third group of minerals have a hexagonal crystalline lattice, and are distinguished only by their chemical composition.

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AGROCHEMISTRY

AVAILABILITY TO PLANT OF PHOSPHATES ADSORBED BY SOIL

By S. N. ALESHIN and E. IGRITSKAIA

(Communicated by D. N. Prianishnikov, Member of the Academy, 29.XII.1937)

Among anions adsorbed by soil, the group of phosphoric acid ions, determining the nutrition of plants (H_2PO_4', HPO_4'', PO_4''') is the most significant for the agricultural chemistry. We are concerned with the study of their adsorbability and of conditions under which they are given up to the plant by the soil. This is the fundamental theme of our works, which initiated in the laboratory of Prianishnikov, as a development of the ideas suggested by him in 1925 (7).

Only a part of these works is reported in the present article, which deals with conditions increasing the availability of phosphate ions to the plants in the presence of anions of other compounds.

As early, as 1934 (2) we succeeded in showing that the availability of phosphates to the plant increased noticeably when organic matter had been introduced into the soil namely anions SiO_3'' and OH' which are easily adsorbed on the surface of colloidal aggregates of soil. The significance of these anions is most actual for soils of the subtropical zone which differ from soils of the temperate zone in their considerable adsorptive capacity with regard to phosphates; therefore, the phosphate fertilizers are efficient there only if applied in excess(1,3).

For that reason the washed off krasnozem (red soil) from Chakva (Adjar Autonomous SSR) tea plantation, was chosen as object of the present research. 60 g of this soil was quite enough for complete adsorption of phosphate ion from 200 cm³ of sodium phosphate solution (pH =6.00) containing 0.2 g of P_2O_5.

The experiment was carried out on soil and sand cultures: the vessels were filled with a mixture of sand and soil and sown with oats. As basic elements of nutrition, were taken (per pot): 0.5 g of K in the form of KNO_3, 0.5 g of N in the form of $Ca(NO_3)_2$, with taking account of nitrogen in potassium fertilizers, 0.2 g of Mg in the form of $MgSO_4$, iron citrate, boron and Mn in small quantities. The fertilizers tested were introduced in following quantities: phosphorus in the form of $Ca(H_2PO_4)_2$—0.2 g (2.84 m mol.) P_2O_5; SiO_3'' in the form of Na_2SiO_3—2.8 m. mol.; OH' in the form of $Ca(OH)_2$ 1.0 g or 13.5 m. mol.; organic matter (4.08 g of humus) in the form of extract from the Kobulety (Adjar Autonomous SSR) peat isolated by sodium hydroxide after preliminary treatment with a 0.1 *n* solution of hydrochloric acid.

The fertilizers investigated were introduced into the soil in a definite order according to the purpose we had set befor us. However, last of them underwent a three-day reciprocal action with 240 g of soil.

In the test-portions of soil, that had been subjected to some preliminary treatment basic fertilizers were introduced and then the rest-portions were intruded into the vessels, containing 5 kg of sand. The oats treated previously with a 0.25% solution of Uspulum were sown on May 31 1935. 21 plants were left for the sake of vegetation. The crop was gathered in a state of full maturity on October 29, 1935.

The scheme of the experiment and the quantities of crops gathered are shown in Table 1.

Table 1

Effect of Anions on Supplying Plant with Phosphate Ions

1st line—crop yield, as a function of P_2O_5 amount given up by the soil;
2nd » —phosphates were introduced the foremost
3rd » — » » » at last;
X designates Co $(OH)_2$, Na_2SiO_3, or organic matter.

Scheme	Control test		$Ca(OH)_2$		Na_2SiO_3		Organic matter	
	Yield in g	Per cent	Yield in g	% to control	Yield in g	% to control	Yield g	% to control
1) Soil + NK	4.4	100	5.5	125	4.4	100	5.8	131
2) » + P + *X* + NK	11.0	100	28.5	258	29.5	267	25.6	232
3) » + *X* + P + NK	—	—	28.8	262	30.1	272	32.3	292
Difference in yield (between 3 and 2)	—	—	0.3	4.0	0.6	5.0	6.7	60

The data tabulated show that in the absence of phosphate ions no significant effect was exerted by tested ions (1st line). The only introduction of phosphates increases the yield, but the effect is more than doubled in the presence of anions competing for a place on the adsorptive surface. Such action may be represented as follows (2nd line):

$$\text{organic matter} < OH' < SiO_3''.$$

The action of these ions was expressed still more distinctly if before the phosphates had been introduced, the soils underwent a reciprocal action with tested ions. In the third line of Table 1 are shown the data of the experiment, which was carried on quite identically with the preceding one, except the order of introduction; namely before the phosphates had been introduced, the 240 g soil samples underwent a reciprocal action with organic matter solution, calcium hydroxide, and sodium silicate respectively. These data show that the order of introduction influences the supply of the plant with phosphate ions which is particularly efficient in the presence of organic matter, protecting the phosphate ions from adsorption on the soil surface.

The effect of the order we followed in applying the above substances manifested itself more strongly in the experiment with organic fertilizers than with the mineral ones. The organic anion produced a greatest crop yield, in comparison with any other combination, when introduced before

phosphate anion fertilizers; and the least yield, in the case of being introduced after phosphates. According to the data of Mattson (4), Demolon and Bastisse (5), the adsorbability of anions may be arranged in a following series:

$$OH' > PO'''_4 > SiO''_3 \text{ organic compounds.}$$

Taking into consideration the above we may note that according to the the yield data of our experiment, there is an inverse dependence between the effect of the order of introduction of anions and their adsorbability. This may be expressed as follows:

$$OH' < Si''_3O < \text{organic compounds.}$$

The longer will continue the reciprocal action of soil with the previously applied fertilizers, the more distinct will be such behaviour, because of the increased tenacity of adsorbed anion.

At the Oxford Congress of Soil Scientists Prof. Wiegner (6) reported that samples of permutite of identical composition, saturated with NH_4^+ and Ca^{2+} successively, and then in inverse succession (the Ca^{2+} ion being introduced first but the whole amount of NH_4^+ and Ca^{2+} being in both cases the same) give back after their treatment with LiCl unequal quantities of NH_4^+ and Ca^{2+}. He was successful in establishing that the ion which had been first adsorbed by permutite was released with greater difficulty. The ions adsorbed at last, were the less tenacious. According to his conception this is accounted for by these ions being situated at the edges and angles of the hard absorbing soil surface.

Evidently, the ideas developed with reference to the adsorption of cations, could be equally applied on the adsorption of anions. From theoretical statements we may infer that the application to soil of organic fertilizers of the type of peat, muck, etc., should give the greatest effect in the case when application of organic fertilizers is foregoing to application of phosphates. Therefore, the farm which has certain stores of such fertilizers can yield greater crop if follows this order of introduction.

We noted above that hydroxile ions are possessing the greatest energy of adsorption. Therefore, we should have to expect the greater effect when supplying the plant with phosphates in the presence of this ion. Taking the quantity of crop yield as a measure of supply we see that such expectation does not justify itself. The same pH value =6.0 was established in all vessels when we started the experiment. During the process of vegetation the pH value changed increasing for the most part to 6.5 and attaining 7.0 only in the experiment with lime. This must be taken into consideration, because it is known that at PH =6.0—6.5 (Teakle, Austin (8)) there begin to form hardly soluble calcium phosphates which influence the supply and diminish the crop yield.

If this statement be true, the best effect might be expected from such an arrangement of the experiment when the limed soils would receive phosphate fertilizers after the introduction of organic fertilizers. This assumption suggested another experiment which we carried on simultaneously with the above with the difference that the three fertilizers tested were introduced in a varying order, with the same three day intervals. In Table 2 the scheme and results of this experiment are shown.

These results confirmed our assumption. We can, therefore, draw some conclusions, important for the agricultural practice: the liming must not take place when the pH value is higher than 6.5; lime and organic fertilizers must immediately precede phosphates. As to silicate ions, such a pre-

Table 2

Effect of Anions on Supplying Plant with Phosphate Ions at the Triple Combination of Fertilizers

The order of introduction / Scheme	PO_4''' + + organic matter		PO_4''' + + organic matter		PO_4''' + + organic matter		Organic matter + PO_4 + X	
	Yield in g	pH	Yield in g	pH	Yield in g	pH	Yield in g	pH
Soil + O + NK	25.6	6.0	—	—	—	—	—	—
» + $Ca(OH)_2$ + NK	—	—	30.8	7.0	34.3	6.9	37.4	6.9
» + Na_2SiO_3 + NK	—	—	32.6	6.6	34.7	6.5	35.4	6.5
X designates $Cu(OH)_2$ or Na_2SiO_3								

caution is not necessary if Na is present, as attendant cation. Besides, this measure was not required by the theory and was not approved by the practice.

Summary

The author investigated the supply of the plant with phosphate ions in the presence of anions which can complete in their adsorbability with phosphate ions.

The experiments showed that the ability of ions to displace PO_4''', estimated after the data on crop yield may be expressed by the following series:

$$SiO_3 > OH' > \text{organic matter.}$$

As the adsorbability of OH is greater than that of SiO_3 (Mattson), this deviation is to be explained as due to formation of hardly soluble $Ca_3(PO_4)_2$.

Further on, it is established that the order of introduction of anions influences the supply of phosphate ions, the greatest effect being attained with organic fertilizers.

Therefore, we have to suppose that lime must precede the introduction of organic matter if the latter is followed by phosphate fertilizers.

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PLANT PHYSIOLOGY

CONCERNING THE PHYSIOLOGY OF THE CRITICAL PERIOD THE LACK OF WATER IN THE SOIL

By F. D. SKASKIN

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Much attention is now paid to the study of the insufficient water supply during the so-called critical periods of the plant growth.

Brownow's systématical investigations (2) carried out during 1897—1916, brought him to the conclusion that plants in some periods of development are especially sensitive to the lack or excess of certain meteorological factors (such as heat, moisture, etc.). These development stages were called by him «critical periods», while factors determining yield were termed «critical factors».

Later on, numerous works of other investigators confirmed the data of Brownow. It was also established that the critical period precedes the formation of the reproductive organs and in cereals, for example, embraces the interval between the stages of stalk formation and earing.

Few works are concerned with the causes of yield drop associated with the drought taking place during critical periods. Some writers [Pulmann (8), 1922, Acci (1), 1932] believe that the plant has an especial need for moisture, when it is ready to a profuse formation of new vegetative organs. Maximow (6) (1926) and Krasnoselskaja-Maximova (4, 5) (1931—1934) believe that when drought occurs during the critical period, water and, possibly nutritious substances are drawn off from the setting organs of fructification by the upper leaves, which intensively exhale vapor. Nossatowsky (7) (1934) came to the conclusion that soil drought, when falls into the period of «earing», lowers the percentage of normally developed flowers and suppresses the processes of fructification. In a paper of 1936 Trubetzkova and Semenova (12) advance the opinion that under conditions of drought during the critical period the upper leaves absorb nitrogenous substances off the ear.

Lately, doubts have been thrown upon the existence of a critical period. For instance Vassiliev (3) (1933), denies the existence of a critical period and takes for granted that the lack of water in the soil during that time can be neutralized by a preliminary hardening of the plant.

To elucidate the question of the possible causes of the yield drop when the latter is connected with the soil drought occurring in the critical period, we carried out our investigations on several varieties of oats, during 1935—1937 (10, 11). The varieties investigated are arranged below in a series according to drought-resistance (the more resistant variety is followed

by a less resistant one): *Avena byzantina* v. *anatolica*, *Avena byzantina* v. *hispanica* (resistant to drought), *Avena sativa* v. *moscovita* 0315 and *Avena sativa* v. *Bäseler* II (not resistant to drought).

Literary data, as well as our studies, showed that the critical period does exist and precedes the formation of reproductive organs while its length varies according to the plant. Most agricultural plants belong to the first group. An insufficient amount of water in the soil during all the periods of development causes a decrease in the yield which is the greater the more it is close to the formation of the reproductive organs (earing or budding). A minority of plants belongs to the second group. Soil drought affects the yield of this group only if occurs close to the formation of the reproductive organs. A decrease of the yield in plants of the second group, due to the lack of water in the soil during the early period of development, is not as great as in those of the first group. Moreover, a certain increase of the yield was shown by some byotipes of the second group when they had been affected by a shallow drought during the early period of development. (In our experiments this phenomenon was observed in oats of the type *Avena byzantina* v. *anatolica*.)

Anatomical and physiological investigations of oat plants affected by the lack of water in the soil during the critical period afforded experimental data which enable us to judge of the possible causes of yield drop under such conditions.

Under lack of water in the soil during this period the greater changes are met with in the structure of halm. In this case the area of the water conducting system is considerably reduced. This phenomenon does not occur under drought during other periods. In some cases (e. g. in Anatolian oats) considerable changes take place in the structure of halm: the mechanical ring disappears, the halm becomes solid; the basic parenchyma gets nearly lignified; many fascicles aquire a concentric appearance or fuse in pairs. It may be supposed that other plants are subject to the same alterations.

When studying the assimilation under a lack of water in the soil we observed that its intensity is abruptly falling during the whole period of the plant development and especially during the critical one. In many cases the assimilation was lost being replaced by an intensive respiration. When watering was renewed the assimilation was rapidly reestablished. In some cases we recorded an intensified assimilation resulting from a renewed watering and lasting several days. However, after a drought during the critical period we have never observed a reestablishment of the assimilation intensity up to the control rate, as it is seen from the table below.

Assimilation the Second—Eighth Day after a Renewed Watering Following Drought in the Critical Period (*Oat Bäseler II;* Data of 1936)

Date of experiment	Assimilation in mg per dm² per hour		Assimilation in % to control Control=100%
	Control	Test	
2. VIII	17.6	2.13	12.2
3. VIII	10.73	6.47	60.86
8. VIII	5.45	2.31	57.70

In some cases (data of 1935) after a renewed watering following drought during the critical period, we observed for a long time the process of respiration even in the drought-resistant Anatolian oats.

While the intensity of assimilation of the control plants (calculated in mg per dm^2 per hour) equalled 1.8; 1.95; 6.35, those watered after the drought showed a respiration which could be expressed by the figures 2.0; 12.33. 6.08 respectively.

In our experiments the amount of carbohydrates under drought conditions was decreased, as a rule, at all the periods of the development of the oats. The only exception was presented by the Anatolian oats in which we could observe, in comparison with the control, a considerable increase of the total carbohydrates (up to 125.7%) as well as of the total soluble sugar (up to 130.2%) if drought occurred at an early period. In this case, there was a conformity between the rise of the figures of the carbohydrates content and the increase of the yield. The lowest carbohydrates content was found after drought during the critical period. The figures of the total carbohydrates fluctuated between 70—80% to the control, while those of the soluble sugar varied within the range of 48—77%. From the above we may infer that, if drought occurs at the critical period, the plant undoubtedly undergoes an acute want of carbohydrates. It may be believed that the decrease of the content of carbohydrates resulted from a considerable decrease in the intensity of the photosynthesis, and possibly, from their being consumed by the respiration process.

It was a matter of course to suppose that the decrease of the amount of carbohydrates must effect the nitrogenous metabolism. We ascertained indeed that the content of protein nitrogen is considerably reduced, under drought during the critical period, while the content of soluble nitrogen is respectively increased. Thus, e. g., in the experiments of 1935, the content of protein nitrogen, in different varieties of oats, varied wihtin the range of 78—89% to the control, while that of the soluble nitrogen rose up to 114.5—163%. Evidently, the hardly changeable forms of nitrogen, such as the protein in the present case, are transformed into mobile forms of soluble nitrogen. Consequently, drought during the critical period involves either a dissolution of albuminous substances or delays their synthesis checked at intermediate soluble products. Hence, we may draw two conclusions: 1) the plant undergoes a lack of substances (i. e., in the present case, of proteins) for the formation of a new protoplasm in the process of their growth and development and in particular for the formation of normally developed reproductive organs, and 2) the accumulated products of the dissolution or unaccomplished synthesis in the form of soluble nitrogen prevent in some way, according to Sabinin (1936) (9) who founded his statement upon the works of Hammet, the normal course of the processes of fructification. In fact, in the course of our investigations a considerable decrease of the amount of normally developed flowers was met with in similar circumstances.

Thus, on the ground of our data we may suppose the behaviour of the plant, under a lack of water during the critical period to bear the following aspect:

The lack of water in the soil during the critical period tells upon the anatomical formation of the plant. The resulting reduction of the area of the water conducting system undoubtedly leads to a decrease in the water supply of the reproductive part being formed.

The abrupt fall in the assimilation process under a lack of water, changed in a number of cases into respiration leading to a considerable decrease of the carbohydrates content in the plant.

A decrease of the carbohydrates content in connection with a reduced content of proteins evidently leads to a want of substances needed for the formation of the reproductive part. As a result of drought during the critical period the panicle is under-developed, the grain becomes small and the yield decreases. It is necessary to record that under the above mentioned circumstances the percentage of normally developed flowers is considerably reduced. The absence of many grains in the panicle points at a partial sterility and depression of the fructification processes. This phenomenon may be only accounted for either by being the result of want of nutrition or by the accumulation of soluble forms of nitrogen hindering the normal course of fructification. This last assumption requires careful checking.

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PLANT ANATOMY

STRUCTURE OF SHRIVELLED («MEAGRE») GRAINS OF HARD WHEAT

By V. G. ALEXANDROV and O. G. ALEXANDROVA

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The question of the «meagreness» of grain in general, and especially of wheat has a great economic importance (1). From a theoretical point of view the phenomenon of meagreness, in all its multiformity, is a most interesting problem; its solution will allow to understand many sides of the life of the grain. However, very little has been done in the domain of the study of shrivelling. For instance, there is no analysis showing with sufficient clearness the difference of structure of a meagre grain or seed, as compared with a fully and normally swollen one.

This paper deals with some results of our research which aimed to display the structural peculiarities that inhere in meagre wheat kernels.

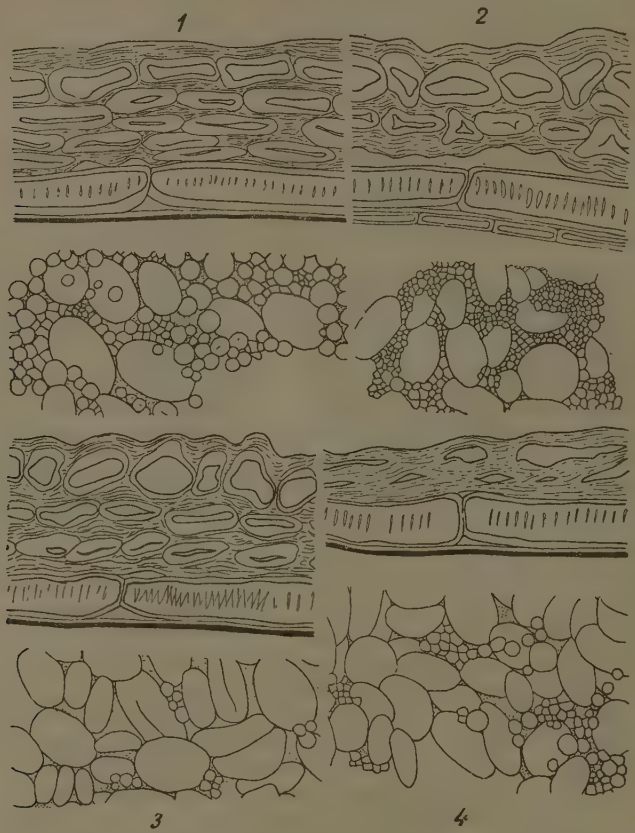
Several causes of the meagreness of wheat kernels are known: dryness of the air and soil, fungus diseases, gliding off water (2), unsufficient ripening previous to drying out, etc. The morphology of the meagre kernels also differs. Shrivelled, too thin, flat, abnormally small kernels, etc., all refer to the category of meagre grain.

In the present report we are stating the result of research made only on kernels of hard wheats. For demonstration we chose the standard sort Hordeiforme 0189 from the Department of geographical cultures of the All-Union Plant Breeding Institute in 1934. This sort possesses a well expressed plasticity, and gives various examples of kernel meagreness. The material we shall examine here is the production of the former Steppe section of All-Union Plant Breeding Institute, Voronej Province, Talovsk Region. In this region, which is a part of the so-called «white spot», there occur yields of wheat with a large percent of meagre grains, the signs of the meagreness being sharply expressed. Our investigations have shown that in hard wheats the structural types of meagreness are more or less equal in the different climates of the USSR, differing only in details.

In comparison with the normally and well developed kernels the eye is struck in the meagre ones by the changes occurring in the structure of the pericarp and in contents of the endosperm cells; therefore, we shall describe several examples of such changes. In our opinion, the changes demonstrated may be considered typical. The inserted five figures of kernel fragments with a different character of meagreness were executed from cuts, made out of the middle part of the kernel; for the pericarp with a seed-coat, from the lateral side of the kernel; and for the endosperm, out of the centre of the lobe. All the preparations were examined in a 10 per cent solution

of sucrose, using Zeiss' objective № 7 and an eye-glass № 12, and drawings were made of them.

On fragment *1* are represented the sections of the coats and endosperm of a fully grown and well swollen kernel. The pericarp consists of six layers of anatomic elements, there individuality being expressed comparatively clearly. Very well are seen the slit-like hills on the transverse cell and the depositions of semi-cellulose in the cavities of the anatomic



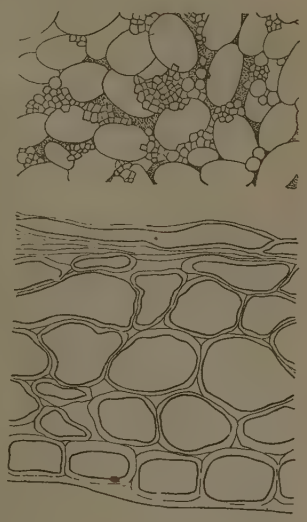
elements of the epi-mesocarp. The mosaic of the endosperm is varied, especially in relation to the chondriosomal starch. The eye is struck by the groups of very minute polyhedral starch granules which closely adhere in some places to the plastid starch granules. The rounded chondriosomal starch granules markedly exceed in number the polyhedral ones, and are much more varied in their dimensions. The difference in size of the chondriosomal starch granules, on one side, and of the plastid ones, on the other, is accentuated with sufficient definitiveness. The kernel is flinty.

Fragment *2* shows the peculiarities of the structure of the very meagre wheat, flattened out along its length, with a floury endosperm. The depositions of the semi-cellulose in the cell cavitities of the anatomic elements of the epi-mesocarp are of the same thickness as in a well swollen kernel, but the number of layers of those elements, instead of being five (fragment *1*)

is lowered to two. The forming of the seed-coat have stopped at one of the intermediate phases: obliteration of the cells is not yet completed. Normally, the seed-coat of a wheat kernel represents a fine film, the remains of the cells forming the latter being hardly distinguishable.

The structure of the central part of the endosperm tissue in the meagre kernel examined is quite other, as compared with a well swollen kernel. The chondriosomal starch is nearly all polyhedral, and it is very small, the interlayers of protein between the starch grains being negligeable.

On fragment *3* is represented a type of the pericarp and endosperm structure in a fully grown and well developed, but strongly shrivelled kernel. The peculiarities of such a kernel are the poorly expressed pores on the wall of the transverse cells, and an insignificant amount of chondriosomal starch. In this relation it stands in contrast with such type of meagre kernel as described above (fragment *2*).



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The majority of plastid starch granules are large; some of them are with longitudinal folds. In a just made up preparation of endosperm tissue of such a kernel the plastid starch granules are strongly wrinkled, each granule representing a small lump of no definite shape. Further the starch granules begin to swell up and smooth out rather rapidly. In normally developed and swollen grains no such changes in the starch granules are observed—all the granules are quite formed into oval or oviform bodies characteristic for wheat. Thus, the plastid starch granules in some types of meagre grain are built up out of less dense and very readily swelling variety of polysaccharide than the starch granules of a normal kernel (similar to dextrines).

Fragment *4* is an example of the structure of an insufficiently developed, small and most meagre kernel. In the epi-mesocarp the cavities of the remnants of two anatomic element layers are only discernable; no depositions of semi-cellulose have been formed. In the endosperm there is but little chondriosomal starch; the polyhedral starch dominates, disposed in separate groups. There are relatively many large plastid starch granules.

In this example of a meagre wheat kernel special attention is attracted by the state of the epi-mesocarp representing a nearly structureless film. Apparently, the factors conditioning the formation of a meagre kernel disturb the coordination of the ferments dissolving the coats of the anatomic elements in the mesocarp. There occurs a fusing of the coats into one common mass, and an obliteration of the cell cavities. The same metamorphosis of the mesocarp tissue is normally brought about in the ripening of the kernels of hooded wheats (spelt, double-grained, monograined, etc.).

On fragment *5* are represented sections of the pericarp and the endosperm of a kernel, as undeveloped and small as on fragment *4*, but not wrinkled. The number of layers of the pericarp remains the same as in a normally developed and swollen kernel (fragment *1*, *6*), but there is no fusing of the seed-coats and no depositions of the semi-cellulose. Therefore in the not grown up and undeveloped grain each cell of the pericarp has preserved well its individual expressiveness, between the cell are dispo-

sed comparatively large intercells. The transverse cells do not show any presence of pores with our method of investigation; besides the very shape of the transverse cells indicates a still not concluded development. The pericarp has all signs of underdevelopment, though the contents of the cells (starch granules abundant in the pericarp cells of young kernels) is absent: neither protoplasm nor nucleus is noticeable in them—the cells are dead.

The structure of the cell content of the endosperm in a meagre kernel of the type represented in fragment *5*, is in general the same, as on fragment *4*, only there is still more of the polyhedral chondriosomal starch, and its domination over the rounded one is still more evident. It is also disposed in groups.

Thus, the manifold variations from the structure of normally developed and well swollen grain one meets with in the meagre kernels of hard wheats, are due to the various causes determining the meagreness of the grain. These structural changes spread over the pericarp as well as over the starchy part of the endosperm, and especially over its central region.

Let us analyse the occuring changes.

During the period of growth of the kernel, preceding its swelling and ripening, only the plastid starch is built up and accumulates in the endosperm. Further on, beginning with the stage of early milk ripening, there appears the chondriosomal starch which fills in the spaces between the plasticid starch ([3, 4]). According to Alexandrov's study ([4]), the chondriosomal starch, arising at the beginning of the process of swelling, is small, homogeneous, polyhedral, and is formed in clusters. Later on, the rounded chondriosomal starch, often greatly varying in size, begins to be deposited, as well as the granules of larger polyhedral chondriosomal starch.

In unmatured kernels (fragment *5*), the minute polyhedral chondriosomal starch formed in clusters stands out especially clearly. The clusters are seen also in the normally grown and swollen kernels (fragment *1*). Whereas, in wrinkled meagre kernels the minute polyhedral chondriosomal starch behaves differently: depending on when the factors conditioning the shrivelling began their action on the forming kernel, and what kind of factors they are, now it accumulates in an exceptionally great amount (fragment *2*, the kernel has dried out at the very beginning of swelling), and then it is nearly absent (fragment *3*, the factors conditioning shrivelling begin to act on the kernel already considerably advanced in swelling). The last case deserves special attention in order to understand the physiology of the wheat kernel. In such a kernel the small chondriosomal polyhedral starch, always arising first in turn, has doubtlessly dissolved, and, as it seems, it has dissolved in order to increase the osmotic pressure in the tissue of the endosperm in counteraction to the influence of the high tension of climatic factors which call forth meagreness.

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DEVELOPMENT MECHANICS

COMPARATIVE DATA ON WING REGENERATION IN BUTTERFLIES

By D. M. STEINBERG

(Communicated by N. V. Nasonov, Member of the Academy, 7.XII.1937)

It has been found for diverse butterfly species that the hypodermal cells in the larvae may regenerate the imaginal discs of the wing. But the investigators who have studied this question have experimented only with the earliest larval stages. It is only in two butterfly species—*Galleria melonella* and *Achroea grisella* that, according to my observations, the regenerative potency of the wing is retained up to the pupal stage, while partial regeneration is possible even during the first two days of the pupal stage.

With the purpose of obtaining evidence regarding the extent of the instability of the hypoderm in other butterfly species, operations on various species were carried out, different larval stages having been chosen for the purpose. In all such experiments a cut was made in the hypoderm under a binocular lens and the imaginal disc of the fore wing, minus the surrounding hypoderm, was extracted. The most detailed investigation respecting the stage in which the larvae lose their wing restoring potency was made on *Pieris brassicae*. Over 100 larvae in stages: II, III, IV and V were operated.

When the imaginal disc has been removed in the II or the III stage the majority of the larvae which stay alive show regeneration of the wing; the wing may vary in size, but it is always quite normal as regards its neuration and design. Removal of the imaginal disc in the IV larval stage produces wing regeneration only if the operation has been carried out in the first half of the period. If the imaginal disc has been removed in the V stage, even if it was done immediately after moulting had taken place, regeneration is never effected. It is to be concluded that in the larvae of *Pieris brassicae* the potency of the hypodermal cells to regenerate the imaginal disc and, consequently, the future wing is retained up to the middle of the IV larval stage; then a critical stage follows after which no regeneration is possible.

In the larvae of *Phylosamia cynthia* no regeneraion occurs if the disc is removed immediately after moulting, which introduces the last stage. The result is the same if the larva in its latest stage is operated in the course of the subsequent days. Of operations carried out on the larvae in their earlier stages we could form no opinion as unfortunately all larvae died.

Removal of the imaginal disc in three *Noctuae* species (*Phytometra gamma*, *Chloridea obsoleta*, *Barathra brassicae*) carried out in the two ultimate larval stages has never produced wing regeneration. The same is true as regards the few operations accomplished on the larvae of *Dicranura vinula*, *Stauropus fagi* and *Acronicta alnae*.

When *Ephestia Kühniella* was operated it was found that larvae of this species, like those of *Galleria* and *Achroea*, are capable of wing regeneration up to their entering the pupal stage; the wing of the hatched butterfly was quite normal, though somewhat smaller in size.

A similar result was obtained with the larvae of *Mecina polygonalis* (Fam. *Pyralididae*). These experiments together with those which were carried out on *Chloridea obsoleta* were made in the Azerbeydjan S. S. R. on the territory of the Saliansk Machine and Tractor Service Station. In these larvae, which feed on astragalus blossoms, the imaginal disc was removed at different periods of the ultimate larval stage. The individuals, which were operated either the day when they entered the pupal stage or earlier, showed a good regeneration of the imaginal disc; of the larvae so operated 18, out of the 19 which had stayed alive, presented in the pupae originated from them as well as in the hatched butterflies a normal wing, though somewhat smaller in size. The larvae, operated one day after the formation of the cocoon showed no regeneration of wing.

The difference in the period of time at which the critical stage permitting of no ulterior regeneration of the wing sets in, is not connected with the degree of differentiation of the imaginal disc. In four species of the Fam. *Pyralididae—G. melonella*, *A. grisella*, *E. Kühniella* and *M. polygonalis*, the disc at the time when the cocoon is produced forms quite a well developed organ possessed of a distinctly developed tracheal system of the future wing. In this developmental stage of the disc its regeneration is still possible.

In *Pieris brassicae* the lacunary and the tracheal systems in the imaginal disc develop at the beginning of the V larval stage, while the critical stage after which regeneration cannot ensue belongs to the IV stage; in this stage the imaginal disc being already fully developed as a two layered sac possess es as yet no lacunary and no tracheal system. In *Vanessa urticae* ([1] [2]) and in *Orgyia antiqua* ([3])—the process takes a similar course. In *Ph. cynthia* the interrelations are different. In this species at the beginning of the ultimate larval stage the disc is still very feebly developed; it reppresents only a simple two layered sac protruding somewhat above the bundles of trachaeoli. Although the disc is in its early stages of development, it is nevertheless already incapable of regeneration. The hypoderm in the larvae of the Fam. *Saturnidae* is evidently highly specialized and is incapable of producing a disc.

In *Noctuae* the imaginal disc differentiates at a very early stage. In the fourth larval stage in the imaginal disc the trachea of the future wing are already distinctly visible. It has so far been impossible to observe wing regeneration in the *Noctuae* group. It occurs only—if at all—in the earliest larval stages. As a matter of fact the primordium of a new imaginal disc may arise in the case of operations carried out on larvae in later stages but such a disc is incapable of differentiating and producing a wing; presumably, because its differentiation normally exacts other physiological conditions, such as characterise the larvae in their earlier stages.

The experimental material which we so far possess leads us to infer that in butterflies the potency of wing regeneration depends at least on two factors. 1) In different species the critical stage in which the hypoderm loses its potency to regenerate a new imaginal disc belongs to different larval stages. 2) The developing imaginal disc must be under the influence of conditions permitting of its normal differentiation and this is possible only if its primordium in non operated larvae makes its appearance and develops relatively late in the course of larval development.

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